SOME LIMIT PROPERTIES OF C-MOVABLY REGULAR CONVERGENCE
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ABSTRACT. Let $d_{\text{C-mov}}^C$ be the metric of C-movably regular convergence for the hyperspace $\mathcal{M}(X)$ of nonempty C-movable subcompacta of a metric space $X$, recently introduced by Cerin. We provide several answers, depending on the choice of property $\alpha$, the class $C$ of topological spaces, and the nature of $A_0$ to the question: If $\lim_{n \to \infty} d_{\text{C-mov}}^C(A_n, A_0) = 0$ and $A_n \in \alpha$ for $n = 1, 2, 3, \ldots$, is it true that $A_0 \in \alpha$? In particular, we are concerned with properties $\alpha$ for which the analogous question, obtained by replacing $d_{\text{C-mov}}^C$ with Borsuk's fundamental metric $d_F$, has an affirmative answer.

1. Introduction. Let $2^X$ denote the collection of nonempty compact subsets of a metric space $(X, d)$. In recent papers, Borsuk [3] and Cerin [8] have defined metrics for (subsets of) $2^X$ that induce stronger topologies than that induced by the well-known Hausdorff metric $d_H$. Several similarities and differences between Borsuk's fundamental metric $d_F$ and Cerin's metric of C-movably regular convergence $d_{\text{C-mov}}^C$ were shown in [8]. Several more similarities and differences are shown in this paper.

There are several properties $\alpha$ for which answers to the following question are known ([3], [5], [9]):

(1.1) If $\lim_{n \to \infty} d_F(A_n, A_0) = 0$ and $A_n \in \alpha$ for $n = 1, 2, 3, \ldots$, is it true that $A_0 \in \alpha$?

For some of these properties, [8] answers the analogous question:

(1.2) If $\lim_{n \to \infty} d_{\text{C-mov}}^C(A_n, A_0) = 0$ and $A_n \in \alpha$ for $n = 1, 2, 3, \ldots$, is it true that $A_0 \in \alpha$?

The answer to (1.1) was shown in [5] to be negative if $\alpha$ is allowed to be an arbitrary hereditary shape property. As pointed out in [9], positive answers to (1.1) can be obtained either by restricting $A_0$ or by taking $\alpha$ to be a particular property. Similarly, (1.2) does not in general have a positive answer, but positive answers are possible when suitable restrictions are placed on $A_0, \alpha$, and the class of topological spaces $C$.

2. Preliminaries. By ANR$(M)$ we mean an absolute neighborhood retract for the
class of metrizable spaces. By ANR we indicate a compact ANR(M). An AR is an
absolute retract for the class of metrizable spaces, compact or not.

A continuous function $f$ is an $e$-map if $f$ has domain and range in a metric space $(X,d)$ with $\sup \{ d(x,f(x)) | x \in X \} < \varepsilon$.

For $A, B \in 2^X$, $X \subseteq M$ where $M \in AR$, $d_{F}(A,B) = \varepsilon$ if $\varepsilon$ is the infimum of those
topology induced.

some

neighborhoods $U$ and $V$ of $A$ and $B$, respectively, in $M$, such that for almost all $k$, $f_k(U)$ and $g_k(V)$ are $e$-maps.

It is shown in [3] that the choice of $M$ is not important: if $h : U_{n=0}^\infty A_n \to N \in AR$

is an embedding, then $\lim_{n \to \infty} d_{E}(A_n,A_0) = 0$ if and only if

Let $C$ be a class of topological spaces. Suppose $M \in ANR(M)$ and $B \subseteq U \subseteq V \subseteq M$

where $U$ and $V$ are neighborhoods of $B$ in $M$. We write $C(U,V,B)$ if for every map

$h : K \to V$ of a member $K$ of $C$ into $V$, if $W$ is a neighborhood of $B$ in $M$ then $f$ is

homotopic in $U$ to a map $g : K \to W$. By [9], $B \subseteq 2^X$ is $C$-movable if for some (hence

every) embedding of $B$ into an ANR(M), for each neighborhood $U$ of $B$ in $M$
there is a neighborhood $V$ of $B$ in $M$ with $V \subseteq U$ such that $C(U,V,B)$. If $C = \{ A \}$ for

some compactum $A$, this coincides with the notion of $A$-movability [1].

The subset of $2^X$ consisting of its $C$-movable members is denoted $mo_{C}(X)$. The
topology induced on $mo_{C}(X)$ by $d_{mo}^{C}$ is characterized in [8] by:

\[ \lim_{n \to \infty} d_{E}(A_n,A_0) = 0 \text{ if and only if} \]

\[ (a) \text{ limit } d_{H}(A_n,A_0) = 0, \text{ and} \]

\[ (b) \text{ for every neighborhood } U \text{ of } A \text{ in } M \text{ there is a neighborhood } V \text{ of } A \text{ in } M \text{ such that } V \subseteq U \text{ and for almost all } n, C(U,V,A_n). \]

As with the topology of $d_{E}$, the choice of $M$ does not affect the topology of

$\lim_{n \to \infty} d_{E}(A_n,A_0) = 0$ if and only if

$\lim_{n \to \infty} d_{H}(A_n,A_0) = 0$, and

for every neighborhood $U$ of $A$ in $M$ there is a neighborhood $V$ of $A$ in $M$
such that $V \subseteq U$ and for almost all $n, C(U,V,A_n)$.

3. Shape domination. Following [2], we say $(U,V)$ is a neighborhood of $(A,B)$
in $(M,N)$ if $U$ is a neighborhood of $A$ in $M$ and $V$ is a neighborhood of $B$ in $N$. If $M$ and

$N$ are AR-spaces, $A \in 2^M$, $B \in 2^N$, and $(U,V)$ is a neighborhood of $(A,B)$ in $(M,N)$,
then $A$ and $B$ are $(U,V)$-equivalent in $(M,N)$ if there are fundamental sequences $f = \{ f_k,A,B \}_{M,N}$ and $g = \{ g_k,B,A \}_{N,M}$ and a neighborhood $(U_0,V_0)$ of $(A,B)$ in $(M,N)$
such that for almost all $k$, $g_k \cdot f_k U_0 \supseteq U_0$, in $U$

the former homotopy holds for almost all $k$, with $U$ replaced by $U_0$. These

notions are weaker than those of shape, respectively.

Our first result strengthens [5, 3.1] and has

\[ (3.1) \text{ THEOREM. Let } X \text{ be a metric space. Suppose } \{ A_n \}_{n=0}^\infty \subseteq 2^X \text{ and } \lim_{n \to \infty} d_{E}(A_n,A_0) = 0. \]

\[ \text{PROOF. Since } A = U_{n=0}^\infty A_n \text{ is compact, } X \text{ is the Hilbert cube. Further, convergence of } \{ F(A_n) \}_{n=1}^\infty \text{ to } F(A_0) \text{ in the topology } \]

it follows that there is no loss of generality in supposing $X = 2$. Let $U$ be a neighborhood of $A_0$ in $Q$. There exists $e > 0$ such that

\[ (3.2) \text{ If } Y \text{ is a topological space and } h,k \in H(X,Y) \text{, then } h = k. \]

Let $N$ be a positive integer such that $n \geq N$. There exist fundamental

\[ \{ g_k,A_0,N \}_{Q,Q'} \subseteq \text{ a neighborhood } (V,W) \text{ of } (f_k,A_0) \text{ such that} \]

\[ (3.3) \text{ } k \geq m \text{ implies } f_k \cdot f_m \cdot g_k \cdot g_m \subseteq e. \]

\[ \text{Also there is a neighborhood } T \text{ of } A_0 \text{ in } Q \text{ with} \]

\[ (3.4) \text{ } V \cup W \subseteq P \text{ and } k \geq m \text{ implies } g_k \cdot f_k \cdot P \subseteq V \cup W \text{ and } \]

\[ \text{It follows from (3.3) and (3.4) that} \]

\[ (3.5) \text{ if } k \geq m \text{ then } U \cup f_k \cdot g_k \cdot W \subseteq P \]

\[ T \cup g_k \cdot f_k (T) \subseteq P \text{ and } d(y,g_k \cdot f_k(y)) < e. \]

\[ \text{By (3.2) and the fact that } P \subseteq U, \text{ it follows that} \]

\[ (3.6) \text{ } k \geq m \text{ implies } f_k \cdot g_k \cdot W \subseteq P. \]

\[ \text{It follows from (3.6) that } A_0 \text{ and } A_0 \text{ are } (U_0,V_0) \text{ equivalent.} \]

An analogue holds for convergence in $T$.

\[ (3.7) \text{ THEOREM. Suppose } X \subseteq M \in ANR(lO). \]

\[ T \subseteq M \text{ is } U \text{ implies } \text{ for almost all } k, f_k \cdot g_k \cdot U_0 \subseteq U_0. \]
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such that for almost all \( k \), \( g_k \cdot f_k \rrequire U \sim I \) in \( U \) and \( f_k \cdot g_k \rrequire V \sim I \) in \( V \). If only the former homotopy holds for almost all \( k \), we say \( A \) is \( U \)-dominated by \( B \) in \( M, N \).

These notions are weaker than those of shape equivalence and shape domination, respectively.

Our first result strengthens [5, 3.1] and has a similar proof.

(3.1) THEOREM. Let \( X \) be a metric space contained in the AR-space \( M \).

Suppose \( \{ A_n \}^{\infty}_{n=0} \subseteq 2^X \) and \( \lim_{n \to \infty} d_F(A_n, A_0) = 0 \). Then for each neighborhood \( U \) of \( A_0 \) in \( M, A_n \) and \( A_0 \) are \( (U, U) \)-equivalent for almost all \( n \).

**PROOF.** Since \( A = \bigcup_{n=0}^{\infty} A_n \) is compact, there is an embedding \( F: A \to Q \), where \( Q \) is the Hilbert cube. Further, convergence of \( \{ A_n \}^{\infty}_{n=1} \) to \( A_0 \) implies the convergence of \( \{ F(A_n) \}^{\infty}_{n=1} \) to \( F(A_0) \) in the topology of \( d_F \). From this and [2, 5.12] it follows that there is no loss of generality in simply assuming that \( M = Q \).

Let \( U \) be a neighborhood of \( A_0 \) in \( Q \). Let \( P \subseteq U \) be a compact ANR neighborhood of \( A_0 \) in \( Q \). There exists \( \epsilon > 0 \) such that

(3.2) If \( Y \) is a topological space and \( h, k: Y \to P \) are maps that are \( \epsilon \)-close, then \( h \approx k \).

Let \( N \) be a positive integer such that \( n \geq N \) implies \( d_F(A_n, A_0) < \epsilon/2 \) and \( A_n \subseteq P \).

Now we fix \( n \geq N \). There exist fundamental sequences \( f = \{ f_k: A_n \to A_n \} \subseteq Q, Q \) and \( g = \{ g_k: A_n \to A_n \} \subseteq Q, Q \), a neighborhood \( (V, W) \) of \( (A_n, A_0) \) in \( (Q, Q) \), and a positive integer \( m \) such that

(3.3) \( k \geq m \) implies \( f_k \rrequire V \) and \( g_k \rrequire W \) are \( \epsilon/2 \)-maps.

Also there is a neighborhood \( T \) of \( A_n \) in \( Q \) with \( T \subseteq V \), and we may assume

(3.4) \( V \cup W \subseteq P \) and \( k \geq m \) implies \( f_k(V) \subseteq P \), \( g_k(W) \subseteq V \), and \( f_k(T) \subseteq W \).

It follows from (3.3) and (3.4) that

(3.5) \( f_k \rrequire (V \cup W) \subseteq P \) and \( d_F(x, f_k(W)) < \epsilon \) for all \( x \in W \); also \( T \cup g_k \rrequire (T \cup W) \subseteq P \) and \( d_F(y, g_k(W)) < \epsilon \) for all \( y \in T \).

By (3.2) and the fact that \( P \subseteq U \), it follows from (3.5) that

(3.6) \( k \geq m \) implies \( 1_W \rrequire f_k \rrequire W \) in \( U \) and \( 1_T \rrequire g_k \rrequire T \) in \( U \).

It follows from (3.6) that \( A_n \) and \( A_0 \) are \( (U, U) \)-equivalent.

An analogue holds for convergence in the topology of \( q_{\text{ANR}} \):

(3.7) THEOREM. Suppose \( X \subseteq M \subseteq \text{ANR}(M) \), \( \{ A_n \}^{\infty}_{n=0} \subseteq \text{mo}_{\text{ANR}}(X) \), and
\[ \lim_{n \to \infty} d_{\text{mo}}^{\text{ANR}}(A_n, A_0) = 0. \] Then for each neighborhood \( U \) of \( A_0 \) in \( M \), \( A_n \) and \( A_0 \) are \( (U, U) \)-equivalent for almost all \( n \).

**Proof.** As in the proof of (3.1), we may assume \( M = Q \).

For a fixed neighborhood \( U \) of \( A_0 \) in \( Q \), let \( V \in \text{ANR} \) be a neighborhood of \( A_0 \) in \( Q \) with \( V \subseteq U \) such that \( \text{ANR}(U, V ; A_0) \). There is a positive integer \( m \) such that \( n \geq m \) implies \( A_n \subseteq V \) and \( \text{ANR}(U, V ; A_n) \).

Let \( n \geq m \) be fixed. There are sequences \( \{V_k\}_k \) and \( \{W_k\}_k \) of neighborhoods of \( A_0 \) and \( A_n \) respectively, in \( Q \) such that \( U = V_0 = W_0 \), \( V = V_1 = W_1 \), \( k \geq 1 \) implies \( V_k \) and \( W_k \) are ANR’s, \( \cap_{k=0}^{\infty} V_k = A_0 \), \( \cap_{k=0}^{\infty} W_k = A_n \), and for all \( k \), \( V_{k+1} \subseteq V_k \), \( W_{k+1} \subseteq W_k \), ANR(\( V_k, V_{k+1} ; A_0 \)), and ANR(\( W_k, W_{k+1} ; A_n \)).

Let \( f_1 = g_1 = 1 \), \( Q' \to Q \). We proceed inductively. Suppose \( 1 \leq k \leq i \) implies we have maps \( f_k, g_k : Q \to Q \) satisfying

\[ (3.8) \quad f_k(V) \subseteq V_k \quad \text{and} \quad g_k(V) \subseteq W_k. \]

Our choices of \( V_k \) and \( W_k \) and (3.8) imply there are maps \( F : V \times I \to V_{i+1} \) and \( G : V \times I \to W_{i+1} \) (I denotes the unit interval) such that \( f_0 = f_i \), \( f_1(V) \subseteq V_{i+1} \), \( G_o = g_i \), \( G_{i+1}(V) \subseteq W_{i+1} \). We obtain maps \( f_{i+1} = g_{i+1} : Q \to Q \) satisfying (3.8) for \( k = i+1 \) by taking \( f_{i+1} \) and \( g_{i+1} \) to be extensions of \( F_1 \) and \( G_1 \), respectively.

We claim \( F = \{f_k, A_n, A_0, A_0\}_k \subseteq Q, Q \) and \( G = \{g_k, A_n, A_0\}_k \subseteq Q, Q \) are fundamental sequences. Let \( P \) be a neighborhood of \( A_0 \) in \( Q \). There is a positive integer \( i \) such that \( V_i \subseteq P \). For \( k \geq i \), \( f_k(V) = f_{k+1}(V) \) in \( V_{k+1} \subseteq V_i \subseteq P \), so \( f \) is a fundamental sequence.

The proof that \( g \) is a fundamental sequence is similar.

It is easily seen that for all \( k \), the following homotopies take place in \( U \):

\[ g_k \cdot f_k | V = g_k \cdot f_k | V = f_k | V = g_k \cdot f_1 | V = g_k | V = g_k | V = f_k | V = f_1 | V = 1. \]

This concludes the proof.

The following is an analogue of [5, 3.11] and a theorem of [9]:

**Theorem.** Let \( X \) be a metric space and let \( \{A_n\}_n \infty \subseteq Q \) be a sequence in \( \text{mo}^{\text{ANR}}(X) \) such that \( \lim_{n \to \infty} d_{\text{mo}}^{\text{ANR}}(A_n, A_0) = 0. \) If \( A_0 \) is calm [7], then \( \text{Sh}(A_n) \supseteq \text{Sh}(A_0) \) for almost all \( n \).

**Proof.** Since the class \( P \) of all finite polyhedra is contained in the class \( \text{ANR} \), it follows from [6, 4.2] that every member of \( \text{mo}^{\text{ANR}}(X) \) is a movable compactum. Since \( A_0 \) is also calm, we have \( A_0 \notin \text{FANR}[9] \).

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Observe that (3.7) implies that if \( U \) is a neighborhood of \( A_0 \) in \( Q \), then \( C \) and \( A_0 \) are U-dominated by \( A_0 \) in \( Q \) for almost all \( n \). It follows that (3.9) is immediate consequence of (3.9):

**Corollary.** If \( C = \text{ANR} \) and \( A_0 \) is an AR-space containing \( B \). From (3.7) it follows that if \( C \) is a compactum in an AR-space \( T \) such that \( \text{Sh}(A_n) \supseteq \text{Sh}(A_0) \) for almost all \( n \).

An immediate consequence of (3.9) is:

**Theorem.** Let \( \{A_n\}_n \infty \subseteq Q \) be a sequence in \( \text{mo}^{\text{ANR}}(X) \) such that \( \lim_{n \to \infty} d_{\text{mo}}^{\text{ANR}}(A_n, A_0) = 0. \) Then \( \text{Sh}(A_n) \supseteq \text{Sh}(A_0) \) for almost all \( n \).

The proof of (3.11) shows that that of \( d_{\text{mo}}^{\text{ANR}}(A_n, A_0) = 0 \).

Clearly a movable compactum is C-movable.

(3.11) that the example of [5], in which finite also provides an example to show the results assumption that \( A_0 \) is calm is dropped. That a Ca is a hereditary shape property, several results properties that are not hereditary shape properties.

For the property \( \alpha \) of being quasi-dominated, a positive answer to (1.1) is [3, 8.1]. Similarly:

**Theorem.** Suppose \( \{A_n\}_n \infty \subseteq Q \) is a sequence in \( \text{mo}^{\text{ANR}}(X) \) such that \( \lim_{n \to \infty} d_{\text{mo}}^{\text{ANR}}(A_n, A_0) = 0. \) Then \( A_n \) is quasi-dominated by \( A_0 \) is quasi-dominated by \( B \).

**Proof.** Since the class \( P \) of all finite polyhedra is contained in the class \( \text{ANR} \), it follows from [6, 4.2] that every member of \( \text{mo}^{\text{ANR}}(X) \) is a movable compactum. Since \( A_0 \) is also calm, we have \( A_0 \notin \text{FANR}[9] \).
Observe that (3.7) implies that if U is a neighborhood of $A_o$ in $Q$ then $A_o$ is U-dominated by $A_n$ in $Q, Q$ for almost all n. It follows from [5, 3.7 and 3.10] that

$$\text{Sh}(A_n) \supseteq \text{Sh}(A_o)$$

for almost all n.

An immediate consequence of (3.9) is:

(3.10) COROLLARY. If $C = \text{ANR}$ and $A_o$ is calm, then (1.2) has a positive answer for every hereditary shape property.

The following theorem of [8] shows that the topology of $d_F$ is stronger than that of $d_{\text{mo}}$:

(3.11) THEOREM. Let $\{A_n\}_{n=0}^{\infty} \subseteq \text{mo} C(X)$. If 
$$\lim_{n \to \infty} d_F(A_n, A_o) = 0,$$
then
$$\lim_{n \to \infty} d_{\text{mo}}(A_n, A_o) = 0.$$

Clearly a movable compactum is C-movable for every class C. It follows from (3.11) that the example of [5], in which finite sets converge to a Cantor set in $d_F$, also provides an example to show the results of (3.9) and (3.10) fail when the assumption that $A_o$ is calm is dropped. That a Cantor set is not calm follows from the fact that a calm compactum has finitely many components [7, 4.1].

4. Passage of $\alpha$ to limits. In this section, we answer (1.2) for several particular choices of $(C, \alpha)$ without making additional assumptions about the nature of $A_o$. In view of (3.11), (1.2) is of interest primarily for properties $\alpha$ for which (1.1) has a positive answer. Although (1.1) was originally stated in [3] under the assumption that $\alpha$ is a hereditary shape property, several results of [9] answer (1.1) positively for properties that are not hereditary shape properties.

For the property $\alpha$ of being quasi-dominated by a fixed compactum $B$ [2], the positive answer to (1.1) is [3, 8.1]. Similarly:

(4.1) THEOREM. Suppose $\{A_n\}_{n=0}^{\infty} \subseteq \text{mo} \text{ANR}(X)$ and 
$$\lim_{n \to \infty} d_{\text{ANR}}(A_n, A_o) = 0.$$ 
If $A_n$ is quasi-dominated by a compactum $B$ for $n = 1, 2, 3, \ldots$, then $A_o$ is quasi-dominated by $B$.

PROOF. Let U be a neighborhood of $A_o$ in an AR-space M containing X. Let P be an AR-space containing B. From (3.7) it follows that there is an $n$ such that $A_o$ is U-dominated by $A_n$ in M. By [2, 7.1] there is a neighborhood V of $A_n$ in M such that if C is a compactum in an AR-space T such that $A_n$ is V-dominated by C in M, T, then $A_o$ is U-dominated by C in M, T. The quasi-domination of $A_n$ by B implies $A_n$ is
V-dominated by \( B \) in M.P. Our choice of \( V \) implies \( A_0 \) is U-dominated by \( B \) in M.P. Since \( U \) is an arbitrary neighborhood of \( A_0 \) in M, it follows that \( A_0 \) is quasi-dominated by \( B \).

Recall that a compactum \( A \) is \( C \)-trivial [9] if for some (hence for every) embedding of \( A \) into \( M \in \text{ANR}(M) \), for each neighborhood \( U \) of \( A \) in \( M \) there is a smaller neighborhood \( V \) of \( A \) in \( M \) such that every map of a member of \( C \) into \( V \) is null-homotopic in \( U \). We will use the following:

\begin{equation}
\text{(4.2) THEOREM. [8]. Let } \{A_n\}_{n=0}^{\infty} \subset 2^X \text{ and let } A_0 \text{ be } C \text{-trivial. Then } A_0 \in \text{mo } C(X), \text{ and if } \lim_{n \to \infty} d_{H}(A_n,A_0) = 0, \text{ then } A_n \in \text{mo } C(X) \text{ for almost all } n \text{ and } \lim_{n \to \infty} d_{\text{mo}}(A_n,A_0) = 0.
\end{equation}

If \( C \) and \( D \) are classes of compacta, \( C \) shape dominates \( D \) if for every \( X \in D \) there is a \( Y \in C \) such that \( Sht(Y) \supset Sht(X) \).

In the following, \( S \) will denote a one-point space, and FAR will denote the class of all compacta that are fundamental absolute retracts.

\begin{equation}
\text{(4.3) LEMMA. Every compactum is FAR-trivial and FAR-movable.}
\end{equation}

PROOF. It is obvious that every compactum is \( \{S\} \)-trivial. Since \( \{S\} \) shape dominates FAR, [6, 3.4] implies every compactum is FAR-trivial.

It follows from the first conclusion of (4.2) that every compactum is FAR-movable.

\begin{equation}
\text{(4.4) COROLLARY. The identity map } i: (2^X, d_H) \to \text{mo FAR}(X) \text{ is a homeomorphism.}
\end{equation}

PROOF. By (4.3), \( i \) is a bijection. Let \( \{A_n\}_{n=0}^{\infty} \subset 2^X \). We must show \( \lim_{n \to \infty} d_H(A_n,A_0) = 0 \) if and only if \( \lim_{n \to \infty} d_{\text{mo}}(A_n,A_0) = 0 \). But this follows from (2.1), (4.2), and (4.3).

\begin{equation}
\text{(4.5) COROLLARY. Let } \alpha_0 \text{ be any property satisfying:}
\end{equation}

(a) if \( A \) is a finite compactum then \( A \in \alpha_0 \) and

(b) there is a compactum \( B \) such that \( B \not\in \alpha_0 \).

Then for \( \alpha = \alpha_0 \) and \( C = \text{FAR}, (1.2) \) has a negative answer.

PROOF. This follows from (4.4), since \( B \) is the limit in \( d_{\text{mo}} \) of a sequence of its finite subsets.

For example, (1.1) has a positive answer if \( \alpha \) is movability [3, 9.1] or nearly.
Since solenoids are neither movable nor nearly 1-movable [10], taking B to be a solenoid in (4.5) yields negative answers for these properties to (1.2). There is a positive answer to (1.1) for the property of zero-dimensionality [9], but by taking B to be any positive-dimensional compactum in (4.5) we get a negative answer to (1.2) for α = zero-dimensionality.

A closed subset B of a metric space X is an a-retract of X [9] if for every e > 0 and every neighborhood U of B in X there is a map r: X → U such that r|B is an e-map. Clearly any retract of X is an a-retract of X. We have:

\[(4.6)\]\[\text{THEOREM. Suppose } B \in 2^X \text{ and } B \in \text{ANR. If } B \text{ is an a-retract of } X\text{ then } B \text{ is a retract of } X.\]

**PROOF.** Since B is a compact ANR, there exist a retraction r of a neighborhood \( U \) of B in X and \( V \) an \( e > 0 \) such that if C is closed in X,

\( f_1, f_2: C \to B \) are \( e \)-close, and \( f_1 \) extends to a map \( F_1: X \to B \), then \( f_2 \) extends to \( F_2: X \to B \).

There is a neighborhood \( U \) of B in X such that \( U \subset V \) and \( r|U \) is an \( e/2 \)-map. Let \( \delta \) satisfy \( 0 < \delta < e/2 \) and \( N e(B) \subset U \).

Since B is an a-retract of X, there is a map \( f: X \to U \) such that \( f|B \) is a \( \delta \)-map. By choice of \( \delta \), \( f(B) \subset U \). It follows that for \( x \in B \), \( d(r \cdot f(x), x) \leq d(r \cdot f(x), f(x)) + d(f(x), x) < e/2 + \delta \leq e \). By taking \( f_1 = r \cdot f|B \), \( f_2 = 1_B \), our choice of \( e \) implies B is a retract of X.

If \( \alpha \) is the property of being an a-retract of an AR \( X \), (1.1) has a positive answer [9]. On the other hand, (1.2) has a negative answer for this property if \( C = \text{FAR} \), as the following shows:

\[(4.7)\]\[\text{EXAMPLE. There is a sequence } \{A_n\}_{n=0}^{\infty} \subset \text{moFAR}(\mathbb{E}^2) \text{ (where } \mathbb{E}^2 \text{ is the euclidean plane) such that } \lim_{n \to \infty} \text{d}_{\text{moFAR}}(A_n, A_0) = 0, A_0 \text{ is not an a-retract of } \mathbb{E}^2, \text{ and } A_n \text{ is an a-retract of } \mathbb{E}^2 \text{ for } n = 1, 2, 3, \ldots.\]

**PROOF.** Let \( S^1 \) be the unit circle in \( \mathbb{E}^2 \) and let \( A_1, A_2, A_3, \ldots \) be a sequence of arcs in \( S^1 \) whose limit in \( d_H \) (and therefore, by (4.5), in \( d_{\text{moFAR}} \)) is \( S^1 \). By (4.6), \( S^1 \) is not an a-retract of \( \mathbb{E}^2 \). However, each of \( A_1, A_2, A_3, \ldots \) is an a-retract of \( \mathbb{E}^2 \).

We remark that (4.5) and (4.7) illustrate how the conclusions of (3.9) and (3.10) fail if the class used does not contain ANR, even when \( A_0 \) is still assumed to be calm.
REFERENCES


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