Some properties of digital covering spaces *

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Abstract

In this paper, we study digital versions of some properties of covering spaces from algebraic topology. Among our results are some that correct or improve upon the presentation of assertions in earlier papers [6, 9, 10].

Key words and phrases: digital image, digital topology, homotopy, fundamental group, covering space, normal subgroup

1 Introduction

Digital Topology has arisen for the study of geometric and topological properties of digital images. In our digital technology, such questions arise in a range of applications, including computer graphics, computer tomography, pattern analysis, and robotic design.

Knowledge of the digital fundamental group is potentially important for Image Analysis, as the fundamental group of a digital image tells us something about the form of the image. Digital fundamental groups of discrete objects have been defined by Kong [13]. Boxer [2] shows how classical methods of Algebraic Topology may be used to construct digital fundamental groups.

In this paper, we use Boxer's digital fundamental groups (defined below in section 2). These groups are defined for digital images of all dimensions and for all choices of adjacency relation, whereas Kong's digital fundamental groups are defined in [13] only in dimension 2 and 3, and only for certain choices of adjacency relation. Note examples are given in [2, 5] for which Boxer's and Kong's fundamental groups are different. However, for the 4-connected adjacency relation in the digital plane, Boxer's and Kong's fundamental groups are the same [2].

The digital covering space is an important tool for computing fundamental groups of digital images. A digital covering space has been introduced by Han [7]. Boxer [5] and Boxer and

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Karaca [6] further develop the theory of digital covering spaces by deriving digital analogs of classical results of algebraic topology, such as results concerning the existence of, and properties of, universal covering spaces. The current paper builds on [6] to expand our knowledge of digital covering spaces; we obtain additional results inspired by analogs in Algebraic Topology [14].

This paper is organized as follows. Section 2 provides some basic notions. In Section 3, we study an action of the fundamental group of the digital image of a digital covering map, on the fiber, the set of points mapping to the base point in the base digital image.

In section 4, we investigate maps between digital coverings. Among the results presented in this section are those concerned with covers $p : (E, e_0) \to (B, b_0)$ for which the automorphism group $A(E, p)$ acts transitively on the fiber $p^{-1}(b_0)$. The example of section 4.2 shows that $A(E, p)$ does not always act transitively on $p^{-1}(b_0)$. Corollary 4.13 and Corollary 4.18 give conditions that are necessary or sufficient for $A(E, p)$ to act transitively on $p^{-1}(b_0)$. Theorem 4.14 is not entirely new, as an incorrect version appeared in [10]. This theorem states that given $(\kappa_0, \kappa_1)$-covering maps $p : (E, e) \to (B, b_0)$ and $q : (E, e_0) \to (B, b_0)$ that are radius 2 local isomorphisms, where $E$ is $\kappa_0$-connected, there is an isomorphism $f : (E, e) \to (E, e_0)$ such that $q \circ f = p$ if and only if $p_*(\pi_1(E, e)) = q_*(\pi_1(E, e_0))$. In Theorem 4.17 and Corollary 4.18, we study the structure of the group $A(E, p)$.

## 2 Preliminaries

Let $\mathbb{Z}$ be the set of integers. Then $\mathbb{Z}^n$ is the set of lattice points in the $n$-dimensional Euclidean space. Let $X \subset \mathbb{Z}^n$ and let $\kappa$ be some adjacency relation for the members of $X$. The pair $(X, \kappa)$ is called a (binary) digital image. When the adjacency relation $\kappa$ is understood, we may refer to $X$ as a digital image without mentioning $\kappa$.

A variety of adjacency relations are used in the study of digital images. Some of the better-known adjacencies are the following.

For a positive integer $l$ with $1 \leq l \leq n$ and two distinct points $p = (p_1, p_2, \ldots, p_n), q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$, $p$ and $q$ are $c_l$-adjacent [4] if

- there are at most $l$ indices $i$ such that $|p_i - q_i| = 1$, and
- for all indices $j$ such that $|p_j - q_j| \neq 1$, $p_j = q_j$.

More general adjacency relations are studied in [11].

A $c_l$-adjacency relation on $\mathbb{Z}^n$ may be denoted by the number of points that are adja-
cent to a point \( p \in \mathbb{Z}^n \). Thus, the \( c_1 \)-adjacency on \( \mathbb{Z} \) may be denoted by the number 2, and \( c_1 \)-adjacent points of \( \mathbb{Z} \) are called 2-adjacent. For example, in the definition of a digital \( \kappa \)-path below, the "2" in the term "\((2, \kappa)\)-continuous function" denotes the \( c_1 \)-adjacency on \( \mathbb{Z} \). Similarly, \( c_1 \)-adjacent points of \( \mathbb{Z}^2 \) are called 4-adjacent; \( c_2 \)-adjacent points of \( \mathbb{Z}^2 \) are called 8-adjacent; and in \( \mathbb{Z}^3, c_1, c_2, \) and \( c_3 \)-adjacent points are called 6-adjacent, 18-adjacent, and 26-adjacent, respectively.

Let \( \kappa \) be an adjacency relation defined on \( \mathbb{Z}^n \). A \( \kappa \)-neighbor of \( p \in \mathbb{Z}^n \) is a point of \( \mathbb{Z}^n \) that is \( \kappa \)-adjacent to \( p \). A digital image \( X \subset \mathbb{Z}^n \) is \( \kappa \)-connected [11] if and only if for every pair of different points \( x, y \in X \), there is a set \( \{x_0, x_1, \ldots, x_r\} \) of points of \( X \) such that \( x = x_0, y = x_r \) and \( x_i \) and \( x_{i+1} \) are \( \kappa \)-neighbors where \( i \in \{0, 1, \ldots, r - 1\} \). A \( \kappa \)-component of a digital image \( X \) is a maximal \( \kappa \)-connected subset of \( X \).

Let \( a, b \in \mathbb{Z} \) with \( a < b \). A digital interval [1] is a set of the form

\[
[a, b]_\mathbb{Z} = \{ z \in \mathbb{Z} | a \leq z \leq b \}.
\]

Let \( X \subset \mathbb{Z}^{n_0} \) and \( Y \subset \mathbb{Z}^{n_1} \) be digital images with \( \kappa_0 \)-adjacency and \( \kappa_1 \)-adjacency respectively. A function \( f : X \to Y \) is said to be \((\kappa_0, \kappa_1)\)-continuous [15, 2], if for every \( \kappa_0 \)-connected subset \( U \) of \( X \), \( f(U) \) is a \( \kappa_1 \)-connected subset of \( Y \). We say that such a function is digitally continuous.

**Proposition 2.1** [15, 2] Let \( X \subset \mathbb{Z}^{n_0} \) and \( Y \subset \mathbb{Z}^{n_1} \) be digital images with \( \kappa_0 \)-adjacency and \( \kappa_1 \)-adjacency respectively. Then the function \( f : X \to Y \) is \((\kappa_0, \kappa_1)\)-continuous if and only if for every pair of \( \kappa_0 \)-adjacent points \( \{x_0, x_1\} \) of \( X \), either \( f(x_0) = f(x_1) \) or \( f(x_0) \) and \( f(x_1) \) are \( \kappa_1 \)-adjacent in \( Y \).

Let \( X \subset \mathbb{Z}^{n_0} \) and \( Y \subset \mathbb{Z}^{n_1} \) be digital images with \( \kappa_0 \)-adjacency and \( \kappa_1 \)-adjacency respectively. A function \( f : X \to Y \) is a \((\kappa_0, \kappa_1)\)-isomorphism [5] (called homeomorphism rather than isomorphism in [1, 2]) if \( f \) is \((\kappa_0, \kappa_1)\)-continuous and bijective and further \( f^{-1} : Y \to X \) is \((\kappa_1, \kappa_0)\)-continuous. A \((\kappa, \kappa)\)-isomorphism \( f : X \to X \) is called a \( \kappa \)-automorphism of \( X \).

For example, let \( X = \mathbb{Z}^2 \) and let \( Y = \{(m, n) \in \mathbb{Z}^2 | m \equiv n \text{ mod } 2\} \). Then the map \( f : X \to Y \) defined by \( f(i, j) = (i + j, i - j) \) is a \((4, 8)\)-isomorphism; its \((8, 4)\)-continuous inverse \( f^{-1} : Y \to X \) is given by

\[
f^{-1}(m, n) = \left(\frac{m + n}{2}, \frac{m - n}{2}\right).
\]

By a digital \( \kappa \)-path of length \( m \) from \( x \) to \( y \) in a digital image \( X \), we mean a \((2, \kappa)\)-continuous function \( f : [0, m]_\mathbb{Z} \to X \) such that \( f(0) = x \) and \( f(m) = y \). If \( f(0) = f(m) \), then \( f \) is called a digital \( \kappa \)-loop, and the point \( f(0) \) is the base point of the loop \( f \). A digital loop \( f \) is said to be
a trivial loop if it is a constant function. A simple closed \( \kappa \)-curve of \( m \geq 4 \) points in a digital image \( X \) is a sequence \( f(0), f(1), \ldots, f(m-1) \) of images of the \( \kappa \)-path \( f : [0, m-1]_\mathbb{Z} \to X \) such that \( f(i) \) and \( f(j) \) are \( \kappa \)-adjacent if and only if \( j = (i \pm 1) \mod m \).

Let \( X \subset \mathbb{Z}^n \) and \( Y \subset \mathbb{Z}^m \) be digital images with \( \kappa_0 \)-adjacency and \( \kappa_1 \)-adjacency respectively. Two \((\kappa_0, \kappa_1)\)-continuous functions \( f, g : X \to Y \) are said to be digitally \((\kappa_0, \kappa_1)\)-homotopic in \( Y \) [2] if there is a positive integer \( m \) and a function \( H : X \times [0, m]_\mathbb{Z} \to Y \) such that

- for all \( x \in X \), \( H(x, 0) = f(x) \) and \( H(x, m) = g(x) \);
- for all \( x \in X \), the induced function \( H_x : [0, m]_\mathbb{Z} \to Y \) defined by
  \[ H_x(t) = H(x, t) \text{ for all } t \in [0, m]_\mathbb{Z}, \]
  is \((2, \kappa_1)\)-continuous; and
- for all \( t \in [0, m]_\mathbb{Z} \), the induced function \( H_t : X \to Y \) defined by
  \[ H_t(x) = H(x, t) \text{ for all } x \in X, \]
  is \((\kappa_0, \kappa_1)\)-continuous.

We say that the function \( H \) is a digitally \((\kappa_0, \kappa_1)\)-homotopy between \( f \) and \( g \).

In [2] Boxer shows that the digitally \((\kappa_0, \kappa_1)\)-homotopy relation is an equivalence relation among digitally continuous functions \( f : (X, \kappa_0) \to (Y, \kappa_1) \).

If \( X \) and \( Y \) are digital images and \( y \in Y \), we denote by \( \tilde{g} \) the constant function \( \tilde{g} : X \to Y \) defined by \( \tilde{g}(x) = y \) for all \( x \in X \).

A pointed digital image is a triple \((X, x_0, \kappa)\), where \((X, \kappa)\) is a digital image and \( x_0 \in X \).

If \((X, x_0, \kappa_0)\) and \((Y, y_0, \kappa_1)\) are pointed digital images, then we write \( f : (X, x_0) \to (Y, y_0) \) to indicate that \( f \) is a function from \( X \) to \( Y \) such that \( f(x_0) = y_0 \).

If \( f : [0, m_1]_\mathbb{Z} \to X \) and \( g : [0, m_2]_\mathbb{Z} \to X \) are digital \( \kappa \)-paths with \( f(m_1) = g(0) \), then define the product \((f \ast g) : [0, m_1 + m_2]_\mathbb{Z} \to X \) [2] by

\[
(f \ast g)(t) = \begin{cases} f(t) & \text{if } t \in [0, m_1]_\mathbb{Z}; \\ g(t - m_1) & \text{if } t \in [m_1, m_1 + m_2]_\mathbb{Z}. \end{cases}
\]

It’s undesirable to restrict loop classes to loops defined on the same digital interval. We have the following notion of trivial extension, which allows a loop to stretch and remain in the same digital class.

Let \( f \) and \( f' \) be \( \kappa \)-loops in a pointed digital image \((X, x_0)\). We say \( f' \) is a trivial extension of \( f \) [2] if there are sets of \( \kappa \)-paths \( \{f_1, f_2, \ldots, f_r\} \) and \( \{F_1, F_2, \ldots, F_p\} \) in \( X \) such that

- \( r \leq p \);
- \( f = f_1 \ast f_2 \ast \cdots \ast f_r \);
- \( f' = F_1 \ast F_2 \ast \cdots \ast F_p \);
• There are indices $1 \leq i_1 < i_2 < \cdots < i_r \leq p$ such that $F_{i_j} = f_j$, $1 \leq j \leq r$; and $i \notin \{i_1, i_2, \ldots, i_r\}$ implies $F_i$ is a trivial loop.

If $f, g : [0, m]Z \to (X, \kappa)$ are $\kappa$-paths such that $f(0) = g(0)$ and $f(m) = g(m)$, then we say that a homotopy $H : [0, m]Z \times [0, M']Z \to X$ between $f$ and $g$ such that for all $t \in [0, M]Z$, $H(0, t) = f(0)$ and $H(m, t) = f(m)$, holds the endpoints fixed (this generalizes a definition of [3]).

Two loops $f, f_0$ with the same base point $x_0 \in X$ belong to the same loop class $[f]_X$ [3] if they have trivial extensions that can be joined by a homotopy that holds the endpoints fixed.

Define $\pi^r_\kappa(X, x_0)$ to be the set of digital homotopy classes of $\kappa$-loops $[f]_X$ in $X$ with base point $x_0$. $\pi^r_\kappa(X, x_0)$ is a group under the product operation defined by $[f]_X \cdot [g]_X = [f \ast g]_X$ (see [2]).

Let $(E, \kappa)$ be a digital image and let $\varepsilon$ be a positive integer. The $\kappa$-neighborhood [7] of $e_0 \in E$ with radius $\varepsilon$ is the set

$$N_\kappa(e_0, \varepsilon) = \{e \in E \mid l_\kappa(e_0, e) \leq \varepsilon\} \cup \{e_0\},$$

where $l_\kappa(e_0, e)$ is the length of a shortest $\kappa$-path from $e_0$ to $e$ in $E$.

The definition of digital covering maps in [7] was simplified in [5] as follows.

**Proposition 2.2** [5] Let $(E, \kappa_0)$ and $(B, \kappa_1)$ be digital images. Then a map $p : E \to B$ is a $(\kappa_0, \kappa_1)$-covering map if and only if both of the following are true:

1. $p$ is a $(\kappa_0, \kappa_1)$-continuous surjection.
2. for each $b \in B$, there exists an indexing set $M$ such that $p^{-1}(b)$ can be indexed as $p^{-1}(b) = \{e_i \mid i \in M\}$ and the following conditions hold:
   - $p^{-1}(N_{\kappa_1}(b, 1)) = \bigcup_{i \in M} N_{\kappa_0}(e_i, 1)$;
   - if $i, j \in M$, $i \neq j$, then $N_{\kappa_0}(e_i, 1) \cap N_{\kappa_0}(e_j, 1) = \emptyset$; and
   - the restriction map $p|_{N_{\kappa_0}(e_i, 1)} : N_{\kappa_0}(e_i, 1) \to N_{\kappa_1}(b, 1)$ is a $(\kappa_0, \kappa_1)$-isomorphism for all $i \in M$.

**Definition 2.3** [8] For $n \in \mathbb{N}$, a $(\kappa_0, \kappa_1)$-covering $p : E \to B$ is a radius $n$ local isomorphism if the restriction map $p|_{N_{\kappa_0}(e, n)} : N_{\kappa_0}(e, n) \to N_{\kappa_1}(b, n)$ is a $(\kappa_0, \kappa_1)$-isomorphism whenever $b \in B$ and $e \in p^{-1}(b)$.

Note that a covering map is a radius 1 local isomorphism, but there are covering maps that are not radius 2 local isomorphisms [5]. A covering map in our sense is essentially a special case of the classical concept of a covering projection onto a graph (see, e.g., chapter 6 of [14]).

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Let \((E, \kappa_0), (B, \kappa_1)\), and \((X, \kappa_2)\) be digital images, let \(p : E \to B\) be a \((\kappa_0, \kappa_1)\)-covering map, and let \(f : X \to B\) be \((\kappa_2, \kappa_1)\)-continuous. A lifting of \(f\) with respect to \(p\) is a \((\kappa_2, \kappa_0)\)-continuous function \(\tilde{f} : X \to E\) such that \(p \circ \tilde{f} = f\) (see [7]).

**Theorem 2.4** [7] Let \((E, \kappa_0)\) be a digital image and \(e_0 \in E\). Let \((B, \kappa_1)\) be a digital image and \(b_0 \in B\). Let \(p : E \to B\) be a \((\kappa_0, \kappa_1)\)-covering map such that \(p(e_0) = b_0\). Then any \(\kappa\)-path \(f : [0, m] \to B\) beginning at \(b_0\) has a unique lifting to a path \(\tilde{f}\) in \(E\) beginning at \(e_0\).

**Theorem 2.5** [8] Let \((E, \kappa_0)\) be a digital image and \(e_0 \in E\). Let \((B, \kappa_1)\) be a digital image and \(b_0 \in B\). Let \(p : E \to B\) be a \((\kappa_0, \kappa_1)\)-covering map such that \(p(e_0) = b_0\). Suppose that \(p\) is a radius 2 local isomorphism. For \(\kappa_0\)-paths \(g_0, g_1 : [0, m] \to E\) starting at \(e_0\), if there is a \(\kappa_1\)-homotopy in \(B\) from \(p \circ g_0\) to \(p \circ g_1\) that holds the endpoints fixed, then \(g_0(m) = g_1(m)\), and there is a \(\kappa_0\)-homotopy in \(E\) from \(g_0\) to \(g_1\) that holds the endpoints fixed.

An example in [5] shows that the conclusion of Theorem 2.5 may not be obtained in the absence of the assumption that \(p\) is a radius 2 local isomorphism.

A digital pointed image \((X, x_0)\) is said to be simply \(\kappa\)-connected [7] if \(X\) is \(\kappa\)-connected and \(\pi_1^\kappa(X, x_0)\) is a trivial group.

In [2] Boxer proves that if \(f : (X, x_0) \to (Y, y_0)\) is a \((\kappa_0, \kappa_1)\)-continuous map of pointed digital images, then \(f_* : \pi_1^\kappa_0(X, x_0) \to \pi_1^\kappa_1(Y, y_0)\), defined by \(f_*([g]) = [f \circ g]\), is a group homomorphism.

From Theorem 2.5, we immediately have the following result.

**Corollary 2.6** [5] Let \((E, \kappa_0)\) be a digital image and \(e_0 \in E\). Let \((B, \kappa_1)\) be a digital image and \(b_0 \in B\). Let \(p : E \to B\) be a \((\kappa_0, \kappa_1)\)-covering map such that \(p(e_0) = b_0\). Suppose that \(p\) is a radius 2 local isomorphism. Then the induced homomorphism \(p_* : \pi_1^\kappa_0(E, e_0) \to \pi_1^\kappa_1(B, b_0)\) is a monomorphism.

The following result describes an algebraic condition that is necessary, and with an additional hypothesis, sufficient for existence of a lifting of a function.

**Theorem 2.7** [5] Let \((E, e_0, \kappa_0)\) and \((B, b_0, \kappa_1)\) be pointed digital images. Let \(p : (E, e_0) \to (B, b_0)\) be a pointed \((\kappa_0, \kappa_1)\)-covering map. Let \(X\) be a \(\kappa_2\)-connected digital image, and let \(x_0 \in X\). Let \(\phi : (X, x_0) \to (B, b_0)\) be a \((\kappa_2, \kappa_1)\)-continuous map of pointed digital images. Consider the following statements.
1. There exists a lifting $\tilde{\phi} : (X, x_0) \to (E, e_0)$ of $\phi$ with respect to $p$.

2. $\phi_* (\pi_1^{\kappa_0} (X, x_0)) \subset \mu_\alpha (\pi_1^{\kappa_0} (E, e_0))$.

Then (1) implies (2). Further, if $p$ is a radius 2 local isomorphism, then (2) implies (1).

3 The action of the group $\pi_1^{\kappa_1} (B, b_0)$ on $p^{-1} (b_0)$

Many of our results are based on analogs in the algebraic topology of Euclidean spaces, as presented, for example, in the section 5.7 of [14].

In this section we study an action of the digital fundamental group of the base digital image of a digital covering map, on the fiber, the set of points mapping to the base point in the base digital image.

Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map that is a radius 2 local isomorphism. To simplify notation, we define $G = \pi_1^{\kappa_1} (B, b_0)$ and $F = p^{-1} (b_0)$. We are going to describe an action of the group $G$ on the set $F$ as a group of permutations. For convenience, the group will act on the right of the set.

Let $e \in F$ and $\alpha \in G$. Represent $\alpha$ by a $\kappa_1$-path $f : [0, m_f] \to B$. Lift $f$ to get a $\kappa_0$-path $g$ in $E$ with $g(0) = e$. Then define the dot operator [14, 9] as a function from $F \times G$ to $F$ by $e \cdot \alpha = g(m_f)$. By Theorem 2.5, this does not depend on the choice of $f \in \alpha$ and so this function is well-defined. Now we derive some properties of this operator. For $e \in F$ and $\alpha, \beta \in G$, we have the following equations (1) and (2), whose proofs are given below (note these properties appear as (4.2) of [9], but proofs are not given there).

$$ e \cdot 1 = e \quad (1) $$

$$ (e \cdot \alpha) \cdot \beta = e \cdot (\alpha \beta) \quad (2) $$

These imply that $\pi_1^{\kappa_1} (B, b_0)$ acts as a group of permutations of $p^{-1} (b_0)$. Equation (1) is valid because the identity element 1 of $G$ is represented by the constant map $\beta_0$. To prove equation (2), lift a $\kappa_1$-loop representing $\alpha$ to a $\kappa_0$-path $f$ from $e$ to $e \cdot \alpha$ and a $\kappa_1$-loop representing $\beta$ to a $\kappa_0$-path $g$ from $e \cdot \alpha$ to $(e \cdot \alpha) \cdot \beta$.

Then $f * g$ is a lift of a path representing $\alpha \beta$, starting at $e$ and ending at $(e \cdot \alpha) \cdot \beta$. Therefore, $(e \cdot \alpha) \cdot \beta = e \cdot (\alpha \beta)$.

**Lemma 3.1** [9] Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map that is a radius 2 local isomorphism. Let $E$ be $\kappa_0$-connected. Then the group $G$ acts transitively on the set $F$, i.e., given $e, e_0 \in F$, there exists $\alpha \in G$ such that $e = e_0 \cdot \alpha$.

For each $e \in F$, let $G_e = \{ \alpha \in G | e \cdot \alpha = e \}$. The set $G_e$ is clearly a subgroup of $G$ and is known as the stabilizer or isotropy group of $e$. 

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Lemma 3.2 Let \( p : (E, e_0) \to (B, b_0) \) be a \((\kappa_0, \kappa_1)\)-covering map that is a radius 2 local isomorphism. Then \( G_{e_0} = p_*(\pi_1^{\kappa_0}(E, e_0)) \).

Proof: Note that \( \alpha \in G_{e_0} \iff \alpha = [f] \) for some \( \kappa_1 \)-path \( f \) that lifts to a \( \kappa_0 \)-loop based at \( e_0 \iff \alpha \in \text{Im}(p_*) \). ■

Theorem 3.3 Let \( p : (E, e_0) \to (B, b_0) \) be a \((\kappa_0, \kappa_1)\)-covering map that is a radius 2 local isomorphism. Let \( E \) be \( \kappa_0 \)-connected. Then there is a bijection between the set \( \pi_1^{\kappa_1}(B, b_0)/p_*(\pi_1^{\kappa_0}(E, e_0)) \) of the right cosets, and the set \( p^{-1}(b_0) \).

Proof: Define the map \( \varphi : G_{e_0} \to F \) taking the right coset \( G_{e_0} \alpha \) to \( e_0 \cdot (G_{e_0} \alpha) = (\text{by equation (2)}) e_0 \cdot \alpha \). By Lemma 3.1, the map \( \varphi \) is a surjection.

Let \( \varphi(G_{e_0} \alpha) = \varphi(G_{e_0} \beta) \) where \( \alpha, \beta \in G \). Then
\[
ee_0 \cdot \alpha = e_0 \cdot \beta \iff e_0 \cdot \alpha \cdot \beta^{-1} = e_0 \iff \alpha \cdot \beta^{-1} \in G_{e_0}
\]
\[
\iff G_{e_0} \alpha = G_{e_0} \beta.
\]
Therefore \( \varphi \) is bijective. ■

Theorem 3.5 Let \( E \) be \( \kappa_0 \)-connected. Let \( p : (E, e_0) \to (B, b_0) \) be a digital covering map and a radius 2 local isomorphism. Then the number of sheets of \( p \) is the index of \( p_*(\pi_1^{\kappa_0}(E, e_0)) \) in \( \pi_1^{\kappa_1}(B, b_0) \).

Proof: This follows from Theorem 3.3. ■

An immediate consequence is the following.

Corollary 3.6 Let \( p : (E, e_0) \to (B, b_0) \) be a \((\kappa_0, \kappa_1)\)-covering map and a radius 2 local isomorphism. If \( E \) is \( \kappa_0 \)-connected and simply \( \kappa_0 \)-connected, then the number of sheets equals to the order of \( \pi_1^{\kappa_1}(B, b_0) \). ■

Lemma 3.7 [6] Let \( h_\alpha : \pi_1^{\kappa_1}(B, b_1) \to \pi_1^{\kappa_1}(B, b_0) \) be defined by
\[
h_\alpha([\beta]) = [\alpha^{-1} \ast \beta \ast \alpha],
\]
where \( \beta \) is a \( \kappa \)-loop at \( b_1 \), \( \alpha : [0, m]_Z \to B \) is a \( \kappa \)-path with \( \alpha(0) = b_1 \) and \( \alpha(m) = b_0 \), and \( \alpha^{-1} \) is the path that reverses \( \alpha \). Then \( h_\alpha \) an isomorphism of groups, ■

Definition 3.4 Let \( p : (E, e_0) \to (B, b_0) \) be a \((\kappa_0, \kappa_1)\)-covering map that is a radius 2 local isomorphism. If \( p^{-1}(b_0) \) has \( n \) elements, then the number \( n \) is called the number of sheets of the digital covering space.
Theorem 3.8 Let $E$ be $\kappa_0$-connected. Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map and a radius 2 local isomorphism. Let $F_0 = p^{-1}(b_0)$, let $b_1 \in B$, and let $F_1 = p^{-1}(b_1)$. Then $|F_0| = |F_1|$. 

Proof: Let $e_1 \in F_1$, and note that $e_0 \in F_0$. Let $\lambda = p \circ \lambda$ be a $\kappa_0$-path in $E$ from $e_0$ to $e_1$. Then $\lambda = p \circ \lambda$ is the corresponding $\kappa_1$-path in $B$ from $b_0$ to $b_1$. By Lemma 3.7, $h_\lambda : \pi_1^0(E, e_0) \to \pi_1^0(E, e_1)$, defined by $h_\lambda([f]) = [\lambda^{-1} \ast f \ast \lambda]$, and $h_\lambda : \pi_1^{\kappa_1}(B, b_0) \to \pi_1^{\kappa_1}(B, b_1)$, defined by $h_\lambda([f]) = [\lambda^{-1} \ast f \ast \lambda]$, are isomorphisms. It follows that $h_\lambda$ induces a bijection between cosets: $\pi_1^{\kappa_1}(B, b_0) : p_* (\pi_1^0(E, e_0)) = \pi_1^{\kappa_1}(B, b_1) : p_* (\pi_1^0(E, e_1))$. Theorem 3.5 now gives the result. ■

4 Deck transformations or automorphisms of the digital covering

4.1 The group $A(E, p)$

In this section, we develop the group of automorphisms of a digital covering space.

Definition 4.1 Let $p : E \to B$ be a digital covering map. A homomorphism [14, 10] of digital covering spaces for $(p, q)$ is a digitally continuous function $f : D \to E$ such that $f(p(d)) = q(d)$ for every $d$ in $D$. If a homomorphism $f : D \to E$ of digital covering spaces for $(p, q)$ is an isomorphism from the digital image $D$ to the digital image $E$, we say $f$ is an isomorphism (of digital covering spaces) for $(p, q)$.

Let $p : E \to B$ be a digital covering map. A deck transformation or automorphism [14, 6] of the digital covering is an isomorphism $h : E \to E$ for $(p, p)$.

Theorem 4.2 [6] If $p : (E, e_0) \to (B, b_0)$ is a pointed $(\kappa_0, \kappa_1)$-covering map and $E$ is $\kappa_0$-connected, then as $e$ ranges over the points of $p^{-1}(b_0)$, $p_* (\pi_1^0(E, e))$ ranges over all conjugates of $p_* (\pi_1^0(E, e_0))$ in $\pi_1^{\kappa_1}(B, b_0)$.

Proposition 4.3 [6] Suppose that $q : (D, d_0) \to (B, b_0)$ is a pointed $(\kappa_0, \kappa_2)$-covering map and $p : (E, e_0) \to (B, b_0)$ is a pointed $(\kappa_1, \kappa_2)$-covering map where $D$ is a $\kappa_0$-connected digital image and $E$ is a $\kappa_1$-connected digital image. If $f, g : D \to E$ are homomorphisms of $(p, q)$ such that $f(d_0) = g(d_0)$, then $f$ and $g$ are identical on $(D, d_0)$. ■

Lemma 4.4 The composition of homomorphisms is a homomorphism; i.e., given coverings $p_i : (E_i, \kappa_i) \to (B, b_0)$ for $i \in \{0, 1, 2\}$, and homomorphisms $h_i : E_i \to E_{i+1}$ with respect to $(p_i, p_{i+1})$ for $i \in \{0, 1\}$, $h_1 \circ h_0 : E_0 \to E_2$ is a homomorphism for $(p_0, p_2)$. 

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Proof: The assertion follows from the observation that for all \( e \in E \),
\[
p_2 \circ (h_1 \circ h_0)(e) = (p_2 \circ h_1) \circ h_0(e) = p_1 \circ h_0(e) = p_0(e). \]

It follows from Lemma 4.4 that the set of all automorphisms of a digital covering space \((E, p)\) is a group under composition of functions. We denote this group by \( A(E, p) \).

4.2 Example: \( A(E, p) \) not acting transitively on \( p^{-1}(b) \)

In section 4.3, we study conditions under which \( A(E, p) \) acts transitively on \( p^{-1}(b) \). Here, we give an example for which \( A(E, p) \) does not act transitively on \( p^{-1}(b) \).

Let \( B = \{ b_i \}_{i=0}^{12} \subseteq (\mathbb{Z}^2, c_2) \), where \( b_0 = (0, 0), b_1 = (1, 1), b_2 = (1, 2), b_3 = (1, 3), b_4 = (0, 4), b_5 = (-1, 3), b_6 = (-1, 2), b_7 = (-1, 1), b_8 = (1, -1), b_9 = (1, -2), b_{10} = (0, -3), b_{11} = (-1, -2), b_{12} = (-1, -1) \). Note we have \( B = B_6 \cup B_8 \), where \( B_6 = \{ b_i \}_{i=0}^{7} \) and \( B_6 = \{ b_0 \} \cup \{ b_i \}_{i=8}^{12} \) are both simple closed \( c_2 \)-curves. See Figure 1.

For \( j \in \mathbb{Z} \), let \( h_j : \mathbb{Z}^2 \to \mathbb{Z}^2 \) be the translation defined by
\[
h_j(x, y) = (x, y + j).
\]

Let \( E \subseteq \mathbb{Z}^2 \) be the set
\[
E = \bigcup_{j \in \mathbb{Z} \setminus \{0\}} (\{6j\} \times \mathbb{Z}) \cup \bigcup_{j \in \mathbb{Z}} h_{6j}(B_8).
\]

Figure 1: The image \( B \). The point labeled \( i \) is \( b_i, i \in \{0, 12\} \)

\[
\cup \bigcup_{j \in \mathbb{Z}} ([\mathbb{Z} \setminus \{0\}] \times \{8j - 1\}).
\]

See Figure 2.
Figure 2: A portion of the image $E$. A point labeled $i$ is mapped by $p$ to $b_i$, $i \in \{0, \ldots, 12\}$.
The adjacency $\kappa$ we use on $E$ will "mostly" be the $c_2$-adjacency. However, we cannot use the $c_2$-adjacency on $E$, as it would cause the map $p$ defined below to be discontinuous. For example, the points $(6, 0)$ and $(7, -1)$ of $E$ are $c_2$-adjacent, but their images under $p$ are $c_2$-disconnected. The $\kappa$-adjacency is defined as follows. Let $x$ and $y$ be distinct points of $E$. Then $x$ and $y$ are $\kappa$-adjacent if and only if

- $x$ and $y$ are $c_2$-adjacent, and
- $x$ and $y$ are not both $c_1$-adjacent to some $p = (6j, 8k - 1) \in E$ for some integers $j, k$ with $j \neq 0$.

Let $e_0 = b_0$. Let $p : (E, e_0) \to (B, b_0)$ be as follows: $p$ wraps each vertical digital line \{6j\} $\times \mathbb{Z}$ in $E$ around $B_8$ such that $p(6j, y) = b_0$ if and only if $y \equiv 7 \bmod 8$; $p$ maps each $h_{8j}(B_8)$ isomorphically to $B_8$ via $h_{8j}^{-1}$; and

$$p(z, 8j - 1) = \begin{cases} b_0 & \text{if } z \neq 0, \quad z \equiv 0 \bmod 6; \\ b_{7+(z \bmod 6)} & \text{otherwise.} \end{cases}$$

Then $p$ is easily seen to be a $(\kappa, c_2)$-covering map. Note \{(0, 0), (6, -1)\} $\subset p^{-1}(b_0)$, but no automorphism $h$ of the digital covering $p$ satisfies $h(0, 0) = (6, -1)$.

This example points out the need to correct the statements of assertions that appeared in [6].

Assertion 4.5 (stated as Corollary 3.5 of [6])

Let $(E, e_0, \kappa_0)$ and $(B, b_0, \kappa_1)$ be pointed digital images. Let $p : (E, e_0) \to (B, b_0)$ be a pointed $(\kappa_0, \kappa_1)$-covering map. Let $X$ be a $\kappa_2$-connected digital image, and let $x_0 \in X$. Let $\phi : (X, x_0) \to (B, b_0)$ be a $(\kappa_2, \kappa_1)$-continuous map of pointed digital images. Let $e \in p^{-1}(b_0)$.

Consider the following statements:

1. There exists a lifting $\tilde{\phi} : (X, x_0) \to (E, e)$ of $\phi$ with respect to $p$.

2. $\phi_*(\pi_1^\kappa (X, x_0))$ is contained in some conjugate of $p_*(\pi_1^\kappa (E, e_0))$.

Then (1) implies (2). Further, if $p$ is a radius 2 local isomorphism, then (2) implies (1).

Assertion 4.6 (stated as Theorem 3.13 of [6])

Suppose that $q : X \to B$ is a digital $(\kappa_0, \kappa_2)$-covering map and $p : E \to B$ is a digital $(\kappa_1, \kappa_2)$-covering map. Let $x_0 \in q^{-1}(b_0)$ and $e_0 \in p^{-1}(b_0)$. Consider the following statements:

1. There is an isomorphism $f : X \to E$ with $f(x_0) = e_0$.

2. $q_*(\pi_1^\kappa (X, x_0))$ and $p_*(\pi_1^\kappa (E, e_0))$ are conjugate in $\pi_1^\kappa (B, b_0)$.

Then (1) implies (2). Further, if $p$ is a radius 2 local isomorphism, then (2) implies (1).

In order to discuss counterexamples to these incorrect assertions, the following is useful.

Lemma 4.7 Let $p : E \to B$ be a $(\kappa_0, \kappa_1)$ covering map. Let $q : D \to B$ be a $(\kappa_2, \kappa_1)$ covering map. Let $f : D \to E$ be a homomorphism for $(p, q)$. If $d$ and $d_0$ are $\kappa_2$-adjacent in $D$, then $f(d)$ and $f(d_0)$ are $\kappa_0$-adjacent.
Proof: Continuity requires that either \( f(d) = f(d_0) \) or that \( f(d) \) and \( f(d_0) \) are \( \kappa_0 \)-adjacent. If the former is true, then, since \( f \) is a homomorphism,

\[
q(d) = pf(d) = pf(d_0) = q(d_0),
\]

contrary to the local isomorphism property of Proposition 2.2. Thus, \( f(d) \) and \( f(d_0) \) must be \( \kappa_0 \)-adjacent. ■

Let us use the previous example, with \( X = E \) and each of \( \kappa_0 \) and \( \kappa_2 \) as the \( \kappa \)-adjacency described above, \( q = p \), and \( \kappa_1 \) as the \( 8 \)-adjacency relation \( c_2 \) on \( B; x_0 = (0,0) = b_0; e = (6,-1) \); and \( e_0 = e \) (unlike the above in which \( e_0 \) was chosen differently). From Theorem 4.2 it follows that \( \rho_*(\pi_1^{\kappa_2}(E,e_0)) \) and \( \phi_*(\pi_1^{\kappa_1}(X,x_0)) \) are conjugate subgroups of \( \pi_1^{\kappa_1}(B,b_0) \), so statement (2) of Assertion 4.5 and of Assertion 4.6 is true. Further, \( p \) is a radius 2 local isomorphism. However, we see the following.

- Statement (1) of Assertion 4.5 does not follow: if there were a lifting \( \tilde{\phi} : (X,x_0) \to (E,e_0) \) of \( \phi \) with respect to \( p \), then the following are easily seen from Lemma 4.7: \( l_{c_2}(\tilde{f}(1,1),\tilde{f}(-1,1)) = 2 \); therefore \( l_{c_2}(\tilde{f}(1,2),\tilde{f}(-1,2)) = 4 \); therefore \( l_{c_2}(\tilde{f}(1,3),\tilde{f}(-1,3)) = 6 \). The latter is impossible, since the continuity of \( \tilde{f} \) requires that \( \tilde{f}(0,4) \) be \( 8 \)-adjacent to each of \( \tilde{f}(1,3) \) and \( \tilde{f}(-1,3) \).

- Statement (1) of Assertion 4.6 does not follow, as, similarly, no isomorphism \( f : X \to E \) can satisfy \( f(x_0) = e_0 \).

In each of the Assertions 4.5 and 4.6, minor modifications of the argument given in “proof” of the incorrect assertion yields a correct assertion. A correct version of Assertion 4.5 (and therefore, of Corollary 3.5 of [6]), is given in the following.

**Corollary 4.8** Let \((E,e_0)\) be a pointed \( \kappa_0 \)-connected digital image. Let \( p : (E,e_0) \to (B,b_0) \) be a pointed \((\kappa_0,\kappa_1)\)-covering map. Let \((X,x_0)\) be a pointed \( \kappa_2 \)-connected digital image, and let \( \phi : (X,x_0) \to (B,b_0) \) be a \((\kappa_2,\kappa_1)\)-continuous map of pointed digital images. Consider the following statements:

1. There is some \( e \in p^{-1}(b_0) \) for which there exists a lifting \( \tilde{\phi} : (X,x_0) \to (E,e_0) \) of \( \phi \) with respect to \( p \).
2. \( \phi_*(\pi_1^{\kappa_2}(X,x_0)) \) is contained in some conjugate of \( p_*(\pi_1^{\kappa_0}(E,e_0)) \).

Then (1) implies (2). Further, if \( p \) is a radius 2 local isomorphism, then (2) implies (1).

A correct version of Assertion 4.6 (and therefore, of Theorem 3.13 of [6]), is given in the following.

**Theorem 4.9** Suppose that \( q : X \to B \) is a digital \((\kappa_0,\kappa_2)\)-covering map and \( p : E \to B \)
is a digital $(\kappa_1, \kappa_2)$-covering map. Let $x_0 \in q^{-1}(b_0)$ and $e_0 \in p^{-1}(b_0)$. Assume $X$ and $E$ are connected. Consider the following statements:

(1) There is an isomorphism $f : (X, x_0) \to (E, e)$ for $(p, q)$ and some $e \in p^{-1}(b_0)$.

(2) $q_*(\pi^\kappa_1(X, x_0))$ and $p_*(\pi^\kappa_1(E, e_0))$ are conjugate in $\pi^\kappa_1(B, b_0)$.

Then (1) implies (2). Further, if $p$ and $q$ are radius 2 local isomorphisms, then (2) implies (1). ■

4.3 Properties of $A(E, p)$

**Proposition 4.10** Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map that is a radius 2 local isomorphism. If $h \in A(E, p)$, $\alpha \in \pi^\kappa_1(B, b_0)$, and $e \in p^{-1}(b_0)$, then $h(e_0) \cdot \alpha = h(e) \cdot \alpha$.

**Proof:** Let $f$ be a $\kappa_1$-loop at $b_0$ representing $\alpha$ and let $g : [0, m_f] \to E$ be a lifting of $f$ with starting point $e$. From the definition we have

$$g(m_f) = e \cdot \alpha.$$ 

The path $h \circ g$ is a lift of $f$ and starts at $h(e)$. Thus it ends at $(h(e)) \cdot \alpha$. On the other hand, it ends at $h(e_0) \cdot \alpha$. ■

**Lemma 4.11** $G_{e_0} \cdot \alpha = \alpha^{-1} G_{e_0} \cdot \alpha$ where $\alpha \in G$.

**Proof:** Notice

$$G_{e_0} \cdot \alpha = \{ \beta \mid (e_0 \cdot \alpha) \cdot \beta = (e_0 \cdot \alpha) \}$$

$$= \{ \beta \mid e_0 \cdot \alpha \cdot \beta \cdot \alpha^{-1} = e_0 \}$$

Thus we have $G_{e_0} \cdot \alpha = \alpha^{-1} G_{e_0} \cdot \alpha$. ■

**Theorem 4.12** Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map that is a radius 2 local isomorphism and let $e \in p^{-1}(b_0)$. Suppose, further, that $E$ is $\kappa_0$-connected. Then the following statements are equivalent:

(1) There is an automorphism $h \in A(E, p)$ such that $h(e_0) = e$.

(2) There is an $\alpha \in N(p_*(\pi^\kappa_1(E, e_0)))$ such that $e = e_0 \cdot \alpha$, where $N(p_*(\pi^\kappa_1(E, e_0)))$ is the normalizer of $p_*(\pi^\kappa_1(E, e_0))$ in $\pi^\kappa_1(B, b_0)$.

(3) $p_*(\pi^\kappa_1(E, e_0)) = p_*(\pi^\kappa_0(E, e))$.

**Proof:** (1) $\Rightarrow$ (3) We see from Definition 4.1 that $h$ is an isomorphism, so $h^{-1}$ exists. Since $h$ is a lifting of $p$, Theorem 2.7 yields

$$p_*(\pi^\kappa_1(E, e_0)) \subset p_*(\pi^\kappa_0(E, e)).$$

Similarly, $h^{-1}$ is a lifting of $p$, so $p_*(\pi^\kappa_0(E, e_0)) \subset p_*(\pi^\kappa_1(E, e_0))$. Statement (3) follows.

(3) $\Rightarrow$ (1) By Theorem 2.7, there is a map $h : E \to E$ lifting $p$ such that $h(e_0) = e$. Then $h$ is a homomorphism of $(p, p)$.

Similarly, there is a homomorphism $h' : E \to E$ of $(p, p)$ such that $h'(e_0) = e_0$. Then $h \circ h'(e) = e$ and $h \circ h'(e_0) = e_0$. By Lemma 4.4 and Proposition 4.3, we have $h \circ h' = 1_E = h' \circ h$. Therefore, $h$ is the desired automorphism.

(2) $\Rightarrow$ (3) If $e = e_0 \cdot \alpha$ for some $\alpha \in N(G_{e_0})$, then $G_{e_0} = \alpha^{-1} G_{e_0} \alpha = (\alpha \cdot \alpha^{-1}) G_{e_0} \alpha = G_{e_0}$, so (3) follows from Lemma 3.2.
(3) $\Rightarrow$ (2) By Lemma 3.2, $G_{e_0} = G_e$. By Lemma 3.1, $e = e_0 \cdot \alpha$ for some $\alpha \in G$. Then $G_{e_0} = G_e = G_{e_0} \cdot \alpha = (\text{by Lemma 4.11}) \alpha^{-1} G_{e_0} \alpha$, so $\alpha \in N(G_{e_0})$. ■

We immediately have the following result.

**Corollary 4.13** Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map that is a radius 2 local isomorphism, where $E$ is $\kappa_0$-connected. Then the subgroup $p_* \pi_1^{e_0}(E, e_0)$ is normal in $\pi_1^{b_0}(B, b_0)$ if and only if the group $A(E, p)$ acts transitively on $p^{-1}(b_0)$.

**Proof:** Suppose that $G_{e_0}$ is normal in $G$. Let $e \in F$. By Lemma 3.1, there exists $\alpha \in G$ such that $e = e_0 \cdot \alpha$. Then $\alpha \in N(G_{e_0})$. By the equivalence of (1) and (2) of Theorem 4.12, there is an automorphism $h \in A(E, p)$ such that $h(e_0) = e$, as desired.

Conversely, assume that $A(E, p)$ acts transitively on $F$: if $e, e_0 \in F$, then there exists $h \in A(E, p)$ with $h(e_0) = e$. Since $p = ph$, it follows that $p_* = p_* h_*$, hence

$$p_*(\pi_1^{e_0}(E, e_0)) = p_* h_*(\pi_1^{e_0}(E, e_0)) = p_* (\pi_1^{e_0}(E, e)).$$

Since $e$ is an arbitrary member of $F$, it follows from Theorem 4.2 that every conjugate of $G_{e_0}$ in $G$ is equal to $G_{e_0}$. Thus $G_{e_0}$ is a normal subgroup of $G$. ■

A version of the following Theorem 4.14, in which our use of “isomorphism” is replaced by “automorphism,” appears as Theorem 6.2 of [10]. Following our proof of Theorem 4.14, we give an example to show that the version in [10] is incorrect.

**Theorem 4.14** Let $e, e_0 \in E$, where $E$ is $\kappa_0$-connected. Let $p : (E, e) \to (B, b_0)$ and $q : (E, e_0) \to (B, b_0)$ be $(\kappa_0, \kappa_1)$ coverings that are radius 2 local isomorphisms. Then there is an isomorphism $f : (E, e) \to (E, e_0)$ such that $q \circ f = p$ if and only if $p_*(\pi_1^{e_0}(E, e)) = q_*(\pi_1^{e_0}(E, e_0))$.

**Proof:** Suppose there is an isomorphism $f : (E, e) \to (E, e_0)$ such that $q \circ f = p$. Then $p_*(\pi_1^{e_0}(E, e)) \subset q_*(\pi_1^{e_0}(E, e_0))$, and the isomorphism $f^{-1}$ exists. Therefore, $q = p \circ f^{-1}$. Thus, $q_*(\pi_1^{e_0}(E, e_0)) \subset p_*(\pi_1^{e_0}(E, e))$, so $p_*(\pi_1^{e_0}(E, e)) = q_*(\pi_1^{e_0}(E, e_0))$.

Conversely, suppose $p_*(\pi_1^{e_0}(E, e)) = q_*(\pi_1^{e_0}(E, e_0))$. By Theorem 2.7, there are liftings $f : (E, e) \to (E, e_0)$ of $p$ with respect to $q$ (so $q \circ f = p$), and $g : (E, e_0) \to (E, e)$ of $q$ with respect to $p$. Then $gf(e) = e$ and $fg(e_0) = e_0$. By Lemma 4.4 and Proposition 4.3, we have $gf = 1_E = fg$. Thus, $f$ is an isomorphism. ■

To show that Han’s version of the previous theorem, in which our use of “isomorphism” is replaced by “automorphism,” is incorrect, consider the following example. Let $B = \{b_i\}_{i=0}^\infty$ be as in Section 4.2 (see Figure 1). Let $p$ and
Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map that is a radius 2 local isomorphism. Define a function $\Theta : N(G_{e_0}) \to A(E, p)$ by $\Theta(\alpha) = h_\alpha$, where $h_\alpha$ is the unique deck transformation such that $h_\alpha(e_0) = e_0 \cdot \alpha$.

By the equivalence $(1) \Leftrightarrow (2)$ of Theorem 4.12, $h_\alpha$ exists. The uniqueness of $h_\alpha$ follows from Proposition 4.3. Thus, $\Theta$ is well defined.

**Theorem 4.17** Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map that is a radius 2 local isomorphism. Suppose, further, that $E$ is $\kappa_0$-connected. The map $\Theta : N(G_{e_0}) \to A(E, p)$ is a group homomorphism and a surjection with kernel $G_{e_0}$. Consequently, we have

$$A(E, p) \approx N(p, \pi^\kappa_1(E, e_0))/p, \pi^\kappa_1(E, e_0).$$

**Proof:** First compute

$$h_\beta \circ h_\alpha(e_0) = h_\beta(e_0 \cdot \alpha) = (by \ Proposition \ 4.10)$$

$$h_\beta(e_0)) \cdot \alpha = (e_0 \cdot \beta) \cdot \alpha = e_0 \cdot (\beta \alpha) = h_{\beta \alpha}(e_0).$$

Thus $\Theta$ is a group homomorphism by Proposition 4.3.

Next note that if $h \in A(E, p)$ then, by Theorem 4.12, there is an $\alpha \in N(G_{e_0})$ such that $h(e_0) = e_0 \cdot \alpha = h_\alpha(e_0)$. By Proposition 4.3, $h = h_\alpha$, which shows that $\Theta$ is onto.
Finally we compute the kernel of $\Theta$:

$$\Theta(\alpha) = h_\alpha = 1 \iff e_0 \cdot \alpha = e_0 \iff \alpha \in G_{e_0}.$$  

By Lemma 3.2, the assertion follows. ■

**Corollary 4.18** Let $p : (E, e_0) \to (B, b_0)$ be a $(\kappa_0, \kappa_1)$-covering map that is a radius 2 local isomorphism, where $E$ is $\kappa_0$-connected.

1. If the map $p$ is regular then

$$A(E, p) \approx \pi^\kappa_1(B, b_0)/p_\ast \pi^\kappa_0(E, e_0).$$

2. If the digital image $E$ is simply $\kappa_0$-connected then $A(E, p) \approx \pi^\kappa_0(B, b_0)$.

**Proof:** (1) This follows from Theorem 4.17 and Corollary 4.13.

(2) Since $E$ is simply $\kappa_0$-connected, $\pi^\kappa_0(E, e_0)$ is a trivial group. Using Corollary 4.13, we see that $p$ is regular. By (1), it follows that $A(E, p) \approx \pi^\kappa_0(B, b_0)$. ■

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