FUNDAMENTAL GROUPS FOR DIGITAL PRODUCTS

LAURENCE BOXER\textsuperscript{a,b} and ISMET KARACA\textsuperscript{c}

\textsuperscript{a}Department of Computer and Information Sciences
Niagara University
NY 14109, U. S. A.

\textsuperscript{b}Department of Computer Science and Engineering
State University of New York at Buffalo, U. S. A.
E-mail: boxer@niagara.edu

\textsuperscript{c}Department of Mathematics
Ege University
Bornova, Izmir 35100, Turkey
E-mail: ismet.karaca@ege.edu.tr

Abstract

We explore conditions under which the fundamental group of a Cartesian product of digital images is isomorphic to the product of the fundamental groups of the factors. Our results are related to assertions that appear in Han's paper [S. E. Han, The $k$-fundamental group of a closed $k$-surface, Inf. Sci. 17 (2007), 3731-3748]; we correct several incomplete and erroneous "proofs" of Han's paper.

1. Introduction

Useful tools from topology for studying properties of a digital image include digital versions of the homotopy type and the fundamental group. Papers including [3-7, 9-12, 14-17, 22] have studied and developed these tools.

The paper [11] proposed an adjacency relation for the Cartesian product of digital images and attempted to study the fundamental group of the resulting...
product image. However, the paper [6] shows that the assertions of [11] concerning the product image are all either incorrect or not correctly proven. In this paper, we explore circumstances under which the "product property" considered in [11] holds for the digital fundamental group. Some of our results duplicate assertions that appear in [12] where there are errors in the associated proofs.

2. Preliminaries

2.1. General properties

Let \( Z \) denote the set of integers. A (binary) digital image is a pair \( (X, k) \), where \( X \subset Z^n \) for some positive integer \( n \) and \( k \) indicates some adjacency relation for the members of \( X \).

Adjacency relations commonly used in the study of digital images in \( Z^n \) include the following [11]. For an integer \( u \) such that \( 1 \leq u \leq n \), we define an adjacency relation as follows. Points

\[
p = (p_1, p_2, \ldots, p_n), \quad q = (q_1, q_2, \ldots, q_n)
\]

are adjacent if

- \( p \neq q \), and
- there are at most \( u \) distinct indices \( i \) for which \( |p_i - q_i| = 1 \), and
- for all indices \( i \), if \( |p_i - q_i| \neq 1 \) then \( p_i - q_i \).

We will call this adjacency the \( c_u \)-adjacency [5], or the \( c_u(n) \)-adjacency when the value of \( n \) requires emphasis. Then \( k = k(u, n) \) is the number of points \( q \in Z^n \) that are adjacent to a given point \( p \) according to this relationship. For example,

- \( k(1, 1) = 2 \),
- \( k(1, 2) = 4; \ k(2, 2) = 8 \);
- \( k(1, 3) = 6; \ k(2, 3) = 18; \ k(3, 3) = 26 \).

More general adjacency relations appear in [13]. In this paper, we are more interested in the value \( u \) of a \( c_u \)-adjacency than in the corresponding value of \( k \).
Let $\kappa$ be an adjacency relation defined on $\mathbb{Z}^n$. A digital image $X \subset \mathbb{Z}^n$ is $\kappa$-connected [13] if and only if for every pair of points $\{x, y\} \subset X, x \neq y$, there exists a set $P = \{x_0, x_1, \ldots, x_c\} \subset X$ of $c + 1$ distinct points such that $x = x_0, x_c = y$, and $x_i$ and $x_{i+1}$ are $\kappa$-adjacent, $i \in \{0, 1, \ldots, c - 1\}$. The set $P$ is a path (see also Definition 2.5, below). We say $c$ is the length of $P$. The following generalizes an earlier definition of [20].

Definition 2.1 [3]. Let $X \subset \mathbb{Z}^{n_0}, Y \subset \mathbb{Z}^{n_1}$. Let $f : X \rightarrow Y$ be a function. Let $\kappa_i$ be an adjacency relation defined on $\mathbb{Z}^{n_i}, i \in \{0, 1\}$. We say $f$ is $(\kappa_0, \kappa_1)$-continuous if the image under $f$ of every $\kappa_0$-connected subset of $X$ is $\kappa_1$-connected.

Definition 2.2 [2]. Let $a, b \in \mathbb{Z}, a < b$. A digital interval is a set of the form

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} | a \leq z \leq b\},$$

in which $c_1$-adjacency is assumed.

For example, if $\kappa$ is an adjacency relation on a digital image $Y$, then the following are equivalent.

- $f : [a, b]_{\mathbb{Z}} \rightarrow Y$ is $(c_1, \kappa)$-continuous.
- For each $c \in [a, b - 1]_{\mathbb{Z}}$, either $f(c) = f(c + 1)$ or $f(c)$ and $f(c + 1)$ are $\kappa$-adjacent.
- For each $[c, d]_{\mathbb{Z}} \subset [a, b]_{\mathbb{Z}}, f([c, d]_{\mathbb{Z}})$ is $\kappa$-connected.

A simple consequence of Definition 2.1, generalizing a similar result of [20], is the following.

Proposition 2.3 [3]. Let $X$ and $Y$ be digital images. Then the function $f : X \rightarrow Y$ is $(\kappa_0, \kappa_1)$-continuous if and only if for every pair of $\kappa_0$-adjacent point $x_0, x_1 \in X$, either $f(x_0) = f(x_1)$, or $f(x_0)$ and $f(x_1)$ are $\kappa_1$-adjacent.

Two digital images $(X, \kappa_X)$ and $(Y, \kappa_Y)$ are $(\kappa_X, \kappa_Y)$-homeomorphic or $(\kappa_X, \kappa_Y)$-isomorphic if there is a one-to-one and onto function $f : X \rightarrow Y$ that is $(\kappa_X, \kappa_Y)$-continuous such that the inverse function $f^{-1} : Y \rightarrow X$ is $(\kappa_Y, \kappa_X)$-continuous [2, 4, 6].
Since a digital image may naturally be considered as a graph whose edges are determined by adjacent pixels, an image homeomorphism is a graph isomorphism. Notice [6] that the term \textit{homeomorphic} may be misleading, in that digital models of homeomorphic Euclidean sets need not be digitally homeomorphic. For example, all Euclidean simple closed curves are homeomorphic, but digital simple closed curves of differing cardinalities are not even of the same digital homotopy types [4]. For the remainder of this paper, we therefore use \textit{isomorphic} rather than \textit{homeomorphic} for digital images.

2.2. Digital homotopy

Intuitively, a homotopy between continuous function \( f, g : X \to Y \) is a continuous deformation of, say, \( f \) over a time period until the result of the deformation coincides with \( g \).

**Definition 2.4** ([3]; see also [14]). Let \( X \) and \( Y \) be digital images. Let \( f, g : X \to Y \) be \((\kappa, \lambda)\)-continuous functions and suppose there is a positive integer \( m \) and a function

\[
F : X \times [0, m] \to Y
\]

such that

- for all \( x \in X \), \( F(x, 0) = f(x) \) and \( F(x, m) = g(x) \);
- for all \( x \in X \), the induced function \( F_x : [0, m] \to Y \) defined by

\[
F_x(t) = F(x, t) \quad \text{for all } t \in [0, m],
\]

is \((c_1, \lambda)\)-continuous;
- for all \( t \in [0, m] \), the induced function \( F_t : X \to Y \) defined by

\[
F_t(x) = F(x, t) \quad \text{for all } x \in X,
\]

is \((\kappa, \lambda)\)-continuous.

Then \( F \) is a digital \((\kappa, \lambda)\)-homotopy between \( f \) and \( g \), and \( f \) and \( g \) are \((\kappa, \lambda)\)-homotopic in \( Y \).

The notation

\[
f \simeq (Y, \kappa, \lambda) g
\]
indicates that functions \( f \) and \( g \) are digitally \((\kappa, \lambda)\)-homotopic in \( Y \). Digital homotopy is an equivalence relation among digitally continuous functions [3, 14]. Further, composition preserves homotopy: If \( f_0, f_1 : X \rightarrow Y \) are \((\kappa, \lambda)\)-continuous functions with \( f_0 \cong (Y, \kappa, \lambda) \ f_1 \) and \( g_0, g_1 : Y \rightarrow Z \) are \((\lambda, \mu)\)-continuous with \( g_0 \cong (Z, \kappa, \mu) \ g_1 \), then \( g_0 \circ f_0 \cong (Z, \kappa, \mu) \ g_1 \circ f_1 \) [3].

2.3. Digital loops

**Definition 2.5** (See [14]). A digital \( \kappa \)-path in a digital image \( X \) is a \((c_1, \kappa)\)-continuous function \( f : [0, m]_Z \rightarrow X \). If, further, \( f(0) = f(m) \), we call \( f \) a digital \( \kappa \)-loop, and the point \( p = f(0) \) is the base point of the loop \( f \). If \( f \) is a constant function, it is called a trivial loop.

If \( f \) and \( g \) are digital \( \kappa \)-paths in \( X \) such that \( g \) starts where \( f \) ends, the product (see [14]) of \( f \) and \( g \), written \( f \star g \), is, intuitively, the \( \kappa \)-path obtained by following \( f \), then following \( g \). Formally, if \( f : [0, m_1]_Z \rightarrow X \), \( g : [0, m_2]_Z \rightarrow X \), and \( f(m_1) = g(0) \), then \( (f \star g) : [0, m_1 + m_2]_Z \rightarrow X \) is defined by

\[
(f \star g)(t) = \begin{cases} 
 f(t), & \text{if } t \in [0, m_1]_Z; \\
 g(t - m_1), & \text{if } t \in [m_1, m_1 + m_2]_Z.
\end{cases}
\]

Restriction of loop classes to loops defined on the same digital interval would be undesirable. The following notion of trivial extension permits a loop to "stretch" within the same pointed homotopy class. Intuitively, \( f' \) is a trivial extension of \( f \) if \( f' \) follows the same path as \( f \), but more slowly, with pauses for rest (subintervals of the domain on which \( f' \) is constant).

**Definition 2.6** [3]. Let \( f \) and \( f' \) be \( \kappa \)-loops in a pointed digital image \((X, x_0)\). We say \( f' \) is a trivial extension of \( f \) if there are sets of \( \kappa \)-paths \( \{f_1, f_2, \ldots, f_k\} \) and \( \{F_1, F_2, \ldots, F_p\} \) in \( X \) such that

1. \( k \leq p \);
2. \( f = f_1 \star f_2 \star \ldots \star f_k \);
3. \( f' = F_1 \star F_2 \star \ldots \star F_p \).

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4. there are indices $1 \leq i_1 < i_2 < \ldots < i_k \leq p$ such that

- $F_{i_j} = f_j, \ 1 \leq j \leq k,$ and

- $i \notin \{i_1, i_2, \ldots, i_k\}$ implies $F_i$ is a trivial loop.

This notion lets us compare the digital homotopy properties of loops whose domains may have differing cardinality, since if $m_1 \leq m_2$, we can obtain [3] a trivial extension of a loop $f : [0, m_1]_Z \rightarrow X$ to $f' : [0, m_2]_Z \rightarrow X$ via

$$f'(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq m_1; \\ f(m_1) & \text{if } m_1 \leq t \leq m_2. \end{cases}$$

A $\kappa$-homotopy $H : [0, M_1]_Z \times [0, M_2]_Z \rightarrow X$ between loops $f, g : [0, M_1]_Z \rightarrow X$ is loop-preserving if for each $t \in [0, M_2]_Z, H(0, t) = H(M_1, t)$. A $\kappa$-homotopy $H : [a, b]_Z \times [0, z]_Z \rightarrow X$ between paths $f, g : [a, b]_Z \rightarrow X$ keeps the endpoints fixed if $H(a, t) = H(a, 0)$ and $H(b, t) = H(b, 0)$ for all $t \in [0, z]_Z$.

**Definition 2.7** ([11], correcting an earlier definition in [4]). Two loops $f_0, f_1$ with the same base point $p \in X$ belong to the same loop class $[f]_X$ if they have trivial extensions that can be joined by a homotopy $H$ that keeps the endpoints fixed.

It was incorrectly asserted as Proposition 3.1 of [4] that the assumption in Definition 2.7, that the homotopy keeps the endpoints fixed, could be replaced by the weaker assumption that the homotopy is loop-preserving; the error was pointed out in [6].

Membership in the same loop class in $(X, x_0)$ is an equivalence relation among digital $\kappa$-loops [3].

We denote by $[f]_X$ the loop class of a loop $f$ in $X$. The next result is used in [3] to show the product operation of our digital fundamental group is well defined.

**Proposition 2.8** [3, 14]. Let $f_1, f_2, g_1, g_2$ be digital loops based at $x_0$ in a pointed digital image $(X, x_0)$, with $f_2 \in [f_1]_X$ and $g_2 \in [g_1]_X$. Then $f_2 \cdot g_2 \in [f_1 \cdot g_1]_X$.
2.4. Digital fundamental group

The digital fundamental group is derived from a classical notion of algebraic topology (see [18, 19, 21]). The version discussed in this section is that developed in [3].

Let \((X, x_0, \kappa)\) be a pointed digital image; i.e., \(X\) is a digital image with adjacency relation \(\kappa\), and \(x_0 \in X\). Define \(\prod_k f(X, x_0)\) to be the set of \(\kappa\)-loop classes \([f]_X\) in \(X\) with base point \(x_0\). By Proposition 2.8, the product operation

\[ [f]_X \cdot [g]_X = [f \ast g]_X, \]

is well defined on \(\prod_k(X, x_0)\); further, the operation \(\cdot\) is associative on \(\prod_k(X, x_0)\) [14].

**Lemma 2.9** [3]. Let \((X, x_0)\) be a pointed digital image. Let \(\bar{x}_0 : [0, m]_Z \to X\) be a trivial loop with image \(\{x_0\}\). Then \(\bar{x}_0\) is an identity element for \(\prod_k(X, x_0)\).

**Lemma 2.10** [3]. If \(f : [0, m]_Z \to X\) represents an element of \(\prod f(X, x_0)\), then the function \(g : [0, m]_Z \to X\) defined by

\[ g(t) = f(m - t) \text{ for } t \in [0, m]_Z \]

is an element of \([f]_X^{-1}\) in \(\prod_k(X, x_0)\).

**Theorem 2.11** [3]. \(\prod_k f(X, x_0)\) is a group under the \(\ast\) product operation, the \(\kappa\)-fundamental group of \((X, x_0)\).

**Definition 2.12** [2, 3, 14]. A digital image \((X, k)\) is \(k\)-contractible if its identity map is \(k\)-homotopic in \(X\) to a constant map, i.e., there exists \(x_0 \in X\) and a digital \(k\)-homotopy \(H : X \times [0, m]_Z \to X\) such that

- \(H(x, 0) = x\) for all \(x \in X\); and
- \(H(x, m) = x_0\) for all \(x \in X\).
If we also have \( H(x_0, t) = x_0 \) for all \( t \in [0, m] \), we say \( X \) is pointed \( k \)-contractible.

The next result implies that the base point may be chosen arbitrarily in a connected digital image for purposes of determining the digital fundamental group.

**Theorem 2.13** [3]. Let \( X \) be a digital image with adjacency relation \( \kappa \). If \( u \) and \( v \) belong to the same \( \kappa \)-component of \( X \), then \( \prod_1^\kappa(X, u) \) and \( \prod_1^\kappa(X, v) \) are isomorphic groups.

**Theorem 2.14** [3]. Suppose \( X \) is a digital image that is pointed \( k \)-contractible. Then \( \prod_1^\kappa(X, x_0) \) is trivial (has one element, the class \([x_0]_X\)).

### 2.5. Digital covering spaces

We use the following.

**Definition 2.15** [11]. Let \((X, k)\) be a digital image and let \( \varepsilon \in \mathbb{N} \). The \( k \)-neighborhood of \( x_0 \in X \) with radius \( \varepsilon \) is the set

\[
N_k(x_0, \varepsilon) = \{x \in X | \ell_k(x_0, x) \leq \varepsilon\} \cup \{x_0\},
\]

where \( \ell_k(x_0, x) \) is the length of a shortest \( k \)-path from \( x_0 \) to \( x \) in \( X \).

Where clarification of the ambient digital image is desirable, we will use the notation \( N_k(x_0, \varepsilon, X) \) for \( N_k(x_0, \varepsilon) \).

A definition of digital covering spaces is given in [11]. In [6], it was shown that this definition can be simplified, as follows.

**Proposition 2.16** [6]. Let \((E, k_0)\) and \((B, k_1)\) be digital images. Let \( p : E \rightarrow B \) be a \((k_0, k_1)\)-continuous surjection. Then the map \( p \) is a \((k_0, k_1)\)-covering map if and only if for each \( b \in B \), there is an index set \( M \) such that

- \( p^{-1}(N_{k_1}(b, 1, B)) = \bigcup_{i \in M} N_{k_0}(e_i, 1, E) \) with \( e_i \in p^{-1}(b) \);
- if \( i, j \in M, i \neq j \), then \( N_{k_0}(e_i, 1, E) \cap N_{k_0}(e_j, 1, E) = \emptyset \); and
- the restriction map \( p|_{N_{k_0}(e_i, 1, E)} : N_{k_0}(e_i, 1, E) \rightarrow N_{k_1}(b, 1, B) \) is a \((k_0, k_1)\)-isomorphism for all \( i \in M \).
Example 2.17 [11]. Let $C \subset \mathbb{Z}^n$ be a simple closed $k$-curve of $m > 3$ points, as realized by a $(c_1, k)$-continuous surjection $f : [0, m - 1] \to C$ such that $f(0)$ and $f(m - 1)$ are $k$-adjacent. Then $(\mathbb{Z}, p, C)$ is a $(c_1, k)$-covering, where the map $p : \mathbb{Z} \to C$ is defined by

$$p(z) = f(z \mod m).$$

Definition 2.18 [10]. For $n \in \mathbb{N}$, a $(k_0, k_1)$-covering $(E, p, B)$ is a radius $n$ local isomorphism if the restriction map $p|_{N_{k_0}(e_1, n, E)} : N_{k_0}(e_1, n, E) \to N_{k_1}(b, n, B)$ is a $(k_0, k_1)$-isomorphism, for all $i \in M$ and all $b \in B$.

Every covering is a radius 1 local isomorphism, by the third property of Proposition 2.16. However, there are coverings that are not radius 2 local isomorphisms [6].

### 2.6. Homotopy lifting

The lifting property is an important property of covering spaces. For digital images $(E, k_0)$, $(B, k_1)$, and $(X, k_2)$, let $p : E \to B$ be a $(k_0, k_1)$ covering map, and let $f : X \to B$ be $(k_2, k_1)$-continuous. A lifting of $f$ (with respect to $p$) is a $(k_2, k_0)$-continuous function $	ilde{f} : X \to E$ such that $p \circ \tilde{f} = f$ [11]. We use the following.

Theorem 2.19 [11]. Let $(E, k_0)$ be a digital image and $e_0 \in E$. Let $(B, k_1)$ be a digital image and $b_0 \in B$. Let $p : E \to B$ be a $(k_0, k_1)$-covering map such that $p(e_0) = b_0$. Then any $k$-path $f : [0, m] \to B$ beginning at $b_0$ has a unique lifting to a path $	ilde{f}$ in $E$ beginning at $e_0$.

It is often useful to lift homotopies, in a fashion modeled on that of Euclidean topology [18].

Theorem 2.20 [10]. Let $(E, k_0)$ be a digital image and $e_0 \in E$. Let $(B, k_1)$ be a digital image and $b_0 \in B$. Let $p : (E, e_0) \to (B, b_0)$ be a pointed $(k_0, k_1)$-covering map. Suppose $p$ is a radius 2 local isomorphism. For $k_0$-paths $g_0, g_1 : [0, m] \to E$ that start at $e_0$, if there is a $k_1$-homotopy in $B$ from $p \circ g_0$ to $p \circ g_1$ that holds the end points fixed, then $g_0(m) = g_1(m)$, and there is a $k_0$-homotopy in $E$ from $g_0$ to $g_1$ that holds the end points fixed.
An example of the need in Theorem 2.20 for the hypothesis that \( p \) is a radius 2 local isomorphism, is given in [6].

In [3] it was shown that if \( f : (X, x_0) \to (Y, y_0) \) is a \((k_0, k_1)\)-continuous map of pointed digital images, then the function \( f_* : \prod_{1}^{k_0} (X, x_0) \to \prod_{1}^{k_1} (Y, y_0) \), defined by \( f_*([g]) = [f \circ g] \), is a group homomorphism.

**Corollary 2.21** [6]. Let \((E, \kappa_0)\) and \((B, \kappa_1)\) be digital images. Let \( e_0 \in E \), \( b_0 \in B \). Let \( p : (E, e_0) \to (B, b_0) \) be a \((\kappa_0, \kappa_1)\)-covering map. If \( p \) is a radius 2 local isomorphism, then the induced homomorphism \( p_* : \prod_{1}^{\kappa_0} (E, e_0) \to \prod_{1}^{\kappa_1} (B, b_0) \) is one-to-one.

### 3. Adjacencies for Cartesian Products

In [11], it is proposed to use the normal product graph adjacency [1] for the Cartesian product. This adjacency is defined as follows. Given points \( x_i, y_i \in (X_i, \kappa_i), i \in \{0, 1\}, (x_0, x_1) \) and \( (y_0, y_1) \) are adjacent in \( X_0 \times X_1 \) if and only if one of the following is satisfied.

- \( x_0 = x_1 \) and \( y_0 \) and \( y_1 \) are \( \kappa_1 \)-adjacent; or
- \( x_0 \) and \( x_1 \) are \( \kappa_0 \)-adjacent and \( y_0 = y_1 \); or
- \( x_0 \) and \( x_1 \) are \( \kappa_0 \)-adjacent and \( y_0 \) and \( y_1 \) are \( \kappa_1 \)-adjacent.

This adjacency has useful properties, but errors in its usage in [11] are shown in [6]. In the following, we will denote by \( \kappa_* \) or \( \kappa_* (\kappa_0, \kappa_1) \) then normal product graph adjacency for the Cartesian product of the digital images \((X_0, \kappa_0)\) and \((X_1, \kappa_1)\).

The following examples show that the normal product graph adjacency often does not coincide with a \( c_u \)-adjacency.

- Suppose \( X_i \subseteq \mathbb{Z}^{n_i} \) for \( i \in \{0, 1\} \) and \( X_i \) has \( c_{m_i} \)-adjacency for some integer \( m_i \) satisfying \( 1 \leq m_i \leq n_i \). If \( a_i \) and \( b_i \) are adjacent in \( X_i \), then \( a_i \) and \( b_i \) may differ in one coordinate. If \( (b_0, b_1) \) may differ in one coordinate, we could have \( X_i = \mathbb{Z}^{n_i} \) and

\[
\begin{align*}
& a_i = (x_i, 1, \ldots, x_i) \in X_0 \times X_1, \\
& b_i = (y_i, 1, \ldots, y_i) \in X_0 \times X_1.
\end{align*}
\]

Therefore, if the \( \delta \) is \( (m_0 + m_1) \), then normal product graph adjacency:

- If \( X_0 \times X_1 \) is \( (a_0, a_1) \)-adjacent and \( (b_0, b_1) \)-adjacent to it, \( c_{m_0} \)-adjacent in \( X_0 \times X_1 \) and \( c_1 \)-adjacent in \( X_1 \) are not adjacent in the normal product graph adjacency.

These example show that normal product graph adjacency

**Proposition**

**Proof.** Let the following are equivalent:

- \((x_0, x_1)\) and \((y_0, y_1)\) are \( \kappa_* \)-adjacent.

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and $b_i$ may differ in as many as $m_i$ coordinates. Thus, the points $(a_0, a_1)$ and $(b_0, b_1)$ may differ in as many as $m_0 + m_1$ coordinates in $Z^{n_0 + n_1}$. For example, we could have $X_i = \{a_i, b_i\}$, where

$$a_i = (x_{i,1}, \ldots, x_{i,n_i}), \quad b_i = (x_{i,1} + 1, x_{i,2} + 1, \ldots, x_{i,m_i} + 1, x_{i,m_i+1}, \ldots, x_{i,n_i}).$$

Therefore, if the Cartesian product $X_0 \times X_1$ uses a $c_u$-adjacency with $u < m_0 + m_1$, then normal product graph adjacency does not imply $c_u$-adjacency.

- If $X_0 \times X_1$ uses $c_u$-adjacency for $u > m_0 + m_1$, there may be points $(a_0, a_1)$ and $(b_0, b_1)$ in $X_0 \times X_1$ that are $c_u$-adjacent but $a_i$ and $b_i$ are not $c_{m_i}$-adjacent in $X_i$. For example, if $X_0 = [0, 1]_Z$ uses $c_1$-adjacency and $X_1 = \{(0, 0), (1, 1)\} \subset Z^2$ uses $c_1$-adjacency, then $(0, 0, 0)$ and $(1, 1, 1)$ are $c_3$-adjacent in $X_0 \times X_1$, although $a_1 = (0, 0)$ and $b_1 = (1, 1)$ are not $c_1$-adjacent in $X_i$. In this case $c_u$-adjacency for $X_0 \times X_1$ does not imply normal product graph adjacency.

- If $m_0 + m_1 < n_0 + n_1$, there may be points $(a_0, a_1)$ and $(b_0, b_1)$ in $X_0 \times X_1$ that are $c_{m_0 + m_1}$-adjacent but not $\kappa_*(c_{m_0}, c_{m_1})$-adjacent. For example, if $X_0 = \{0\} \subset Z$ uses $c_1$-adjacency and $X_1 = \{(0, 0), (1, 1)\} \subset Z^2$ using $c_1$-adjacency, then $(0, 0, 0)$ and $(0, 1, 1)$ are $c_2$-adjacent in $X_0 \times X_1$, but not $\kappa_*(c_1, c_1)$-adjacent, since $(0, 0)$ and $(1, 1)$ are not $c_1$-adjacent in $X_1$.

These examples show that he most promising situation for coincidence of the normal product graph adjacency and $c_u$-adjacency is $(X_i, c_{n_i}) \subset Z^{n_i}, i \in \{0, 1\}$, and $u = n_0 + n_1$. We have the following.

**Proposition 3.1.** For digital images $(X_i, c_{n_i}) \subset Z^{n_i}, i \in \{0, 1\}$, the normal product graph adjacency coincides with the $c_{n_0 + n_1}$-adjacency for $X_0 \times X_1$.

**Proof.** Let $(x_i, y_i) \subset X_i, i \in \{0, 1\}$. Suppose $(x_0, y_0) \neq (x_1, y_1)$. The following are equivalent.

- $(x_0, x_1)$ and $(y_0, y_1)$ are $\kappa_*(c_{n_0}, c_{n_1})$-adjacent.
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- \( x_0 \) and \( y_0 \) are identical or \( c_{n_0} \)-adjacent, and \( x_1 \) and \( y_1 \) are identical or \( c_{n_1} \)-adjacent.

- In all \( n_0 \) coordinates in \( Z^{n_0} \), \( x_0 \) and \( y_0 \) differ by an absolute value of at most 1, and in all \( n_1 \) coordinates in \( Z^{n_1} \), \( x_1 \) and \( y_1 \) differ by an absolute value of at most 1.

- In all \( n_0 + n_1 \) coordinates in \( Z^{n_0+n_1} \), \((x_0, y_0)\) and \((x_1, y_1)\) differ by an absolute value of at most 1.

- \((x_0, y_0)\) and \((x_1, y_1)\) are \( c_{n_0+n_1} \)-adjacent.

In order to have a product property for the fundamental groups of digital images, i.e., in order to have

\[
\prod_1 \gamma (X_0 \times Y_0, (x_0, y_0)) = \prod_1 \gamma (X_0, x_0) \times \prod_1 \gamma (X_1, x_1),
\]

where \((X_i, x_i, \kappa_i)\) are pointed connected digital images, \( i \in \{0, 1\} \), it is necessary that \( \kappa \) be appropriately related to \( \kappa_0 \) and \( \kappa_1 \); when such a relationship does not exist, the product property may fail, as shown by the following example. MSC\(_8 \subset Z^2 \) [11] is a simple closed \( c_2 \)-curve that is not a \( c_1 \)-curve (see Figure 1). Let \( p_0 = (-1, -2) \in MSC_8 \). Notice the two \( c_2 \)-neighboring points

\[
\begin{array}{|c|c|c|c|c|}
\hline
& -3 & -2 & -1 & 0 & 1 \\
\hline
-3 & \bullet & \bullet & \bullet & \bullet & \bullet \\
-2 & \bullet & \bullet & \bullet & \bullet & \bullet \\
-1 & \bullet & \bullet & \bullet & \bullet & \bullet \\
0 & \bullet & \bullet & \bullet & \bullet & \bullet \\
1 & \bullet & \bullet & \bullet & \bullet & \bullet \\
\hline
\end{array}
\]

Figure 1. MSC\(_8 \) – a digital simple closed \( c_2 \)-curve from [11].

of \( p_0 \) in MSC\(_8 \) are not \( c_1 \)-neighbors of \( p_0 \). Therefore, \( \{(p_0, p_0)\} \) is a \( c_1 \)-component of MSC\(_8 \times MSC_8 \) in \( Z^4 \). Since the fundamental group of a pointed digital image is that of the component of the base point [3], it follows by Theorem 2.14 that \( \prod_1 \gamma (MSC_8 \times MSC_8, (p_0, p_0)) = \{0\} \). However, since

\[
\prod_1 \gamma (MSC_8, p_0) = \{0\}
\]

defined by

\[
p = p_0
\]

is \( (\kappa, (\kappa_0, \kappa_1), \kappa_0, \kappa_1) \).

Proof. Given \( \alpha = c \) or \( c \) and \( c \) are continuous. We have

Proposition 4.2.

\[
p_0 : (E_0, e_0) \rightarrow (X_0, e)
\]

\[
p_1 : (E_1, e_1) \rightarrow (X_1, e)
\]

\[
p = p_0
\]

or \( p(a, b) \) and \( p(b, a) \) are continuous.

Corollary 4.2.

\[
p_0 : (E_0, e_0) \rightarrow (X_0, e)
\]

\[
p_1 : (E_1, e_1) \rightarrow (X_1, e)
\]

\[
p = p_0
\]

is \( (\kappa, (\kappa_0, \kappa_1), \kappa_0, \kappa_1) \).
\[ \prod_{1}^{g} (\text{MSC}_g, \ p_0) \equiv \mathbb{Z} [6, 11, 15], \text{we have} \]
\[ \prod_{1}^{g2} (\text{MSC}_g, \ p_0) \times \prod_{1}^{g2} (\text{MSC}_g, \ p_0) \equiv \mathbb{Z}^2. \]

4. Product Property

In order for the product of covering maps to be a covering map, it must be continuous. We have the following.

**Proposition 4.1.** Suppose, for \(i \in \{0, 1\}\), \((E_i, e_i)\) are pointed digital images with \(\kappa_i\)-adjacency; \((X_i, x_i)\) are pointed digital images with \(\lambda_i\)-adjacency; and \(p_i : (E_i, e_i) \rightarrow (X_i, x_i)\) is a digital covering map. Then the product map
\[ p = p_0 \times p_1 : (E_0 \times E_1, (e_0, e_1)) \rightarrow (X_0 \times X_1, (x_0, x_1)) \]
defined by
\[ (p_0 \times p_1)(a, b) = (p_0(a), p_1(b)), \]
is \((\kappa_*(\kappa_0, \kappa_1), \kappa_*(\lambda_0, \lambda_1))\)-continuous.

**Proof.** Given \(\kappa_*(\kappa_0, \kappa_1)\)-adjacent points \((a, b)\) and \((c, d)\) in \(X_0 \times X_1\), \(a = c \) or \(a\) and \(c\) are \(\kappa_0\)-adjacent; and \(b = d\) or \(b\) and \(d\) are \(\kappa_1\)-adjacent. Since \(p_0\) and \(p_1\) are continuous, \(p_0(a) = p_0(c)\) or \(p_0(a)\) and \(p_0(c)\) are \(\lambda_0\)-adjacent; and \(p_1(b) = p_1(d)\) or \(p_1(b)\) and \(p_1(d)\) are \(\lambda_1\)-adjacent. Therefore, either
\[ p(a, b) = (p_0(a), p_1(b)) = (p_0(c), p_1(d)) = p(c, d), \]
or \(p(a, b)\) and \(p(c, d)\) are \(\kappa_*(\lambda_0, \lambda_1)\)-adjacent. Thus, \(p\) has the asserted continuity.

**Corollary 4.2.** Suppose, for \(i \in \{0, 1\}\), \((E_i, e_i)\) are pointed digital images with \(\kappa_i\)-adjacency; \((X_i, x_i)\) are pointed digital images with \(\lambda_i\)-adjacency; and \(p_i : (E_i, e_i) \rightarrow (X_i, x_i)\) is a digital covering map. Then the product map
\[ p = p_0 \times p_1 : (E_0 \times E_1, (e_0, e_1)) \rightarrow (X_0 \times X_1, (x_0, x_1)) \]
is a \((\kappa_*(\kappa_0, \kappa_1), \kappa_*(\lambda_0, \lambda_1))\)-covering map.
Proof. This follows from Proposition 2.16 and Proposition 4.1.

Let $S(d_i, n_i)$ be a digital simple closed $c_{n_i}$-curve in $\mathbb{Z}^{n_i}$ with $d_i = |S(d_i, n_i)|$, $i \in \{0, 1\}$. Let $S(d_i, n_i) = \{s_{i, j}\}_{j=0}^{d_i-1}$, where $s_{i, j}$ and $s_{i, k}$ are $c_{n_i}$-adjacent if and only if $j = k + 1 \mod d_i$ or $k = j + 1 \mod d_i$. By Example 2.17, the function $p_i : \mathbb{Z} \rightarrow S(d_i, n_i)$ defined by

$$p_i(x) = s_{i, x \mod d_i}$$

is a $(c_1, c_{n_i})$-covering map.

Lemma 4.3. Let $S(d_i, n_i) \subset \mathbb{Z}^{n_i}$ be a digital simple closed $c_{n_i}$-curve such that

$$S(d_i, n_i) \neq N_{c_{n_i}}(x, 2, S(d_i, n_i)) \text{ for all } x \in S(d_i, n_i), i \in \{0, 1\}.$$

Then the covering map

$$p = p_0 \times p_1 : (\mathbb{Z}^2, c_2) \rightarrow (S(d_0, n_0) \times S(d_1, n_1), c_{n_0+n_1})$$

is a radius 2 local isomorphism.

Proof. We easily see the following.

- Given $(a, b) \in (\mathbb{Z}^2, c_2)$,

$$N_{c_2}((a, b), 2, \mathbb{Z}^2) = [(a - 2, a + 2) \mathbb{Z} \times [b - 2, b + 2] \mathbb{Z}].$$

- We have

$$N_{c_{n_0+n_1}}(p(a, b), 2, S(d_0, n_0) \times S(d_1, n_1))$$

$$= N_{c_{n_0+n_1}}((p_0(a), p_1(b)), 2, S(d_0, n_0) \times S(d_1, n_1))$$

$$= N_{c_{n_0}}(p_0(a), 2, S(d_0, n_0)) \times N_{c_{n_1}}(p_1(b), 2, S(d_1, n_1))$$

$$= p_0([(a - 2, a + 2) \mathbb{Z} \times p_1([(b - 2, b + 2) \mathbb{Z]) = p(N_{c_2}((a, b), 2, \mathbb{Z}^2)).$$

Since $S(d_i, n_i) \neq N_{c_{n_i}}(x, 2, S(d_i, n_i))$ for all $x \in S(d_i, n_i)$, it follows easily that $p$ is a radius 2 local isomorphism.
The following is suggested by, but differs from, Theorem 5.5(2) of [12].

**Theorem 4.4.** Consider the map \( p_1 \times p_2 : \mathbb{Z} \times \mathbb{Z} \to SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2} \) given by \( p_1 \times p_2 (t_1, t_2) = (x_{t_1} \mod l_1, y_{t_2} \mod l_2) \), where \( SC_{k_1}^{n_1, l_1} = \{ x_i \}_{i=0}^{l_1-1} \subset \mathbb{Z}^n_1 \) such that \( x_i \) and \( x_j \) are \( k_1 \)-adjacent if and only if \( i = j \pm 1 \mod l_1 \) and \( SC_{k_2}^{n_2, l_2} = \{ y_i \}_{i=0}^{l_2-1} \subset \mathbb{Z}^n_2 \) such that \( y_i \) and \( y_j \) are \( k_2 \)-adjacent if and only if \( i = j \pm 1 \mod l_2 \) or \( i = j \mod l_2 \). Then \( p_1 \times p_2 \) is a \((c_2, c_{n_1+n_2})\)-covering map if \( k_i = c_{n_i}, i \in \{1, 2\} \).

In Han's version, the hypothesis is that \( k_i = c_1 \) and the conclusion is that \( p_1 \times p_2 \) is a \((c_1, c_1)\)-covering. The conclusion of a \((c_2, c_{n_1+n_2})\)-covering is better suited to our use in proving Theorem 4.7. Also, our version is more interesting, since a major issue in the question of whether a map is a covering is the question of continuity. That \( p_1 \times p_2 \) is \((c_1, c_1)\)-continuous is an easy consequence assuming that each of \( p_1 \) and \( p_2 \) is \((c_1, c_1)\)-continuous.

**Proof of Theorem 4.4.** This follows from Proposition 3.1 and Corollary 4.2.

From Theorem 2.19, Theorem 2.20, Corollary 4.2, and Lemma 4.3, if \( g_0 \) and \( g_1 \) are \( c_{n_0+n_1} \)-loops in \( S(d_0, n_0) \times S(d_1, n_1) \) based at \( (s_0, s_1) \) that represent the same member of \( \prod_{1}^{c_{n_0+n_1}} (S(d_0, n_0) \times S(d_1, n_1), (s_0, s_1)) \), then there are lifts \( \tilde{g}_0, \tilde{g}_1 \) of trivial extensions of \( g_0, g_1 \), respectively, into \( \mathbb{Z}^2 \), each starting at \((0, 0)\) and ending at the same point in \((p_0 \times p_1)^{-1} (s_0, s_1) = d_0 \mathbb{Z} \times d_1 \mathbb{Z} \).

Thus, we consider the function

\[
\Psi : d_0 \mathbb{Z} \times d_1 \mathbb{Z} \to \prod_{1}^{c_{n_0+n_1}} (S(d_0, n_0) \times S(d_1, n_1), (s_0, s_1))
\]

defined as follows. Given \((d_0 x, d_1 y) \in d_0 \mathbb{Z} \times d_1 \mathbb{Z}, \) let \( f_{x, y} : [0, d_0 x + d_1 y] \mathbb{Z} \to \mathbb{Z}^2 \) be the \((c_1, c_2)\)-continuous function

\[
f_{x, y}(t) = \begin{cases} (t, 0) & \text{if } 0 \leq t \leq d_0 x; \\ (d_0 x, t - d_0 x) & \text{if } d_0 x \leq t \leq d_0 x + d_1 y. \end{cases}
\]
Note the path \( f_{x,y} \) terminates at \((d_0 x, d_1 y)\). Then we define \( \Phi \) by

\[
\Phi(d_0 x, d_1 y) = [p \circ f_{x,y}].
\]

Theorem 4.7, below, appears as Corollary 5.8 of [12]. However, Han's argument on behalf of this assertion uses an unproven assertion that

\[
p^{-1}(b_0) \equiv N(p_*(\prod_1^{K_0} (E, e_0)))/p_*(\prod_1^{K_0} (E, e_0)) \tag{1}
\]

if \( p^{-1}(b_0) \) is a group, where \( p : (E, e_0) \rightarrow (B, b_0) \) is a \((\kappa_0, \kappa_1)\)-covering map and \( N(p_*(\prod_1^{K_0} (E, e_0))) \) is the normalizer of \( p_*(\prod_1^{K_0} (E, e_0)) \) as a subgroup of \( \prod_1^{K_0} (B, b_0) \) (see [12], (4.3), p. 3737, used in the "proof" of that paper's Theorem 4.4). This assertion (1) appears to be incorrect, in light of the following.

**Theorem 4.5.** Let \( p : (E, e_0) \rightarrow (B, b_0) \) be a \((\kappa_0, \kappa_1)\)-covering map that is a radius 2 isomorphism. If \( E \) is \( \kappa_0 \)-connected and \( p^{-1}(b_0) \) is a group, then

\[
\prod_1^{K_1} (B, b_0)/p_*(\prod_1^{K_0} (E, e_0)) \equiv p^{-1}(b_0).
\]

In order to prove this assertion, we recall the following from [9]. The action of the group \( \prod_1^{K_1} (B, b_0) \) on \( p^{-1}(b_0) \) is defined as follows. Given \( e_0 \in p^{-1}(b_0) \), \( \alpha \in \prod_1^{K_1} (B, b_0) \), define \( e_0 \cdot \alpha \) to be the terminal point of a lifting of a representative of \( \alpha \) starting at \( e_0 \); by Theorem 2.20, this terminal point is well defined. It is shown in [9] (this is also stated, but not proven, in the earlier paper [12]) that for all \( \alpha, \beta \in \prod_1^{K_1} (B, b_0) \),

\[
(e_0 \cdot \alpha) \cdot \beta = e_0 \cdot (\alpha \beta). \tag{2}
\]

**Proof of Theorem 4.5.** Define the map \( \varphi : \prod_1^{K_1} (B, b_0)/p_*(\prod_1^{K_0} (E, e_0)) \rightarrow p^{-1}(b_0) \) as follows. We use the abbreviation \( G_{e_0} \) for \( p_*(\prod_1^{K_0} (E, e_0)) \). For \( \alpha \in \prod_1^{K_1} (B, b_0) \), \( \varphi(G_{e_0} \alpha) = e_0 \cdot \alpha \).
It is shown in [9] that $\varphi$ is a bijection. Now, observe that

$$\varphi((G_{e_0} \alpha) \times (G_{e_0} \beta)) = \varphi(G_{e_0} (\alpha \beta)) = e_0 \cdot (\alpha \beta) = (\text{by equation (2)})$$

$$(e_0 \cdot \alpha) \cdot \beta = \varphi(G_{e_0} \alpha) \times \varphi(G_{e_0} \beta).$$

Therefore, $\varphi$ is a bijective homomorphism, hence an isomorphism.

The following is excerpted from Theorem 3.1 of [7].

**Theorem 4.6.** Suppose $(X, \kappa)$ is a digital image such that $X = \bigcup_{j=1}^{\infty} X_j$, where for all $j$ we have

- $X_j \subseteq X_{j+1}$;
- $X_j$ is bounded;
- $i_j : X_j \hookrightarrow X_{j+1}$ is the inclusion map $i_j(x) = x$.

If the induced homomorphisms $(i_j)_* : \prod_1^\kappa(X_j) \rightarrow \prod_1^\kappa(X_{j+1})$ are isomorphisms, then the inclusion map $i_X : X_1 \hookrightarrow X$ induces an isomorphism $(i_X)_* : \prod_1^\kappa(X_1) \rightarrow \prod_1^\kappa(X)$.

In the following theorem, the restriction to non-contractible digital simple closed curves does not greatly limit applicability, as it is shown in [8] that a $\kappa$-simple closed curve $S$ is not $\kappa$-contractible if $|S| > 4$.

**Theorem 4.7.** Let $S(d_i, n_i)$ be a digital simple closed $c_{n_i}$-curve in $\mathbb{Z}^n_i$ that is not $c_{n_i}$-contractible in $\mathbb{Z}^{n_i}$, such that $S(d_i, n_i) \neq N_{c_{n_i}}(x, 2, S(d_i, n_i))$ for all $x \in S(d_i, n_i), i \in \{0, 1\}$. Then

$$\prod_{i=1}^{c_{n_0}+n_1} (S(d_0, n_0) \times S(d_1, n_1), (s_0, s_1)) = \mathbb{Z}^2.$$

**Proof.** By Proposition 3.1 and Corollary 4.2, the product map $p_0 \times p_1$ is a $(c_2, c_{n_0}+n_1)$-covering map, and by Lemma 4.3, this map is a radius 2 local isomorphism. Since $\mathbb{Z} \times \mathbb{Z}$ is $c_2$-connected, we have from Theorem 4.5 that...
(using \( p = p_1 \times p_2 \) and \((B, b_0) = (S(d_0, n_0) \times S(d_1, n_1), (s_0, s_1))\)

\[
p^{-1}(b_0) \equiv \prod_{1}^{k_1}(B, b_0)/p_1 \left( \prod_{1}^{k_0}(E, e_0) \right),
\]
or

\[
Z \times Z \equiv d_0 Z \times d_1 Z = p^{-1}(b_0)
\]

\[
\equiv \prod_{1}^{c_{n_0+n_1}}(B, b_0)/p_1 \left( \prod_{1}^{c_2}(Z^2, (0, 0)) \right)
\]

\[
\equiv \prod_{1}^{c_{n_0+n_1}}(B, b_0),
\]

the latter isomorphism since \( Z^2 \) is \( c_2 \)-simply connected (which follows easily from Theorem 4.6, using \( Z^2 = \bigcup_{i=1}^{\infty} [0, i]_Z \times [0, i]_Z \)).

5. Further Remarks

We have shown that for non-contractible digital simple closed curves \((S_i, c_{n_i}) \subset Z^{n_i}, i \in \{0, 1\}\), the digital fundamental group has the "product property",

\[
\prod_{1}^{c_{n_0+n_1}}(S_0 \times S_1, (s_0, s_1)) \equiv \prod_{1}^{c_{n_0}}(S_0, s_0) \times \prod_{1}^{c_{n_1}}(S_1, s_1)
\]

We have shown that for the purpose of obtaining a product property, the natural adjacency relations to use are the \( c_{n_i} \)-adjacencies for \( S_i \subset Z^{n_i} \) and the \( c_{n_0+n_1} \)-adjacency for \( S_0 \times S_1 \subset Z^{n_0+n_1} \); indeed, we have shown that other adjacencies for the Cartesian product need not yield an analogous result.

References


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